

How Do Non-Reproductive Groups Affect Population Growth?

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Abstract

We describe several models of population dynamics, both unstructured and gender-structured, that include groups of individuals who do not reproduce. We analyze the effect that the non-reproductive group may have on the dynamics of the whole population in terms of the vital rates and the proportion of non-reproductive individuals, and we provide specific examples for real populations.

1 Introduction

The basic dynamics of a population depends on the interplay between reproduction and mortality. Long-term trends of the population growth or decline are usually not affected by *temporary* changes in reproduction or mortality, but rather by long-term or *permanent* changes in these vital dynamics parameters. Examples of the latter are the extremely significant reduction in mortality during the XXth century triggered by discoveries such as penicillin, and the reduction in the mean number of progeny per woman that accompanied social conquests by women. One factor whose long-term effect on population growth has not been examined is the segregation of portions of the reproductive population into non-reproductive classes, such as same-sex isolation groups—like prisons, religious groups, or life-long homosexual groups— or other non-reproductive classes—who by choice or medical reasons remain childless for life. Under normal circumstances, childless individuals who are incarcerated for the entirety of their reproductive years will have no sexual contact with the opposite gender and, therefore, remain childless for life. Similarly, men and women who participate exclusively in homosexual relationships (or in none at all) during all their lives will remain childless. Some of these groups (religious or incarcerated individuals isolated during all their reproductive years) are quite small; others (lifelong homosexuals and individuals who choose or are forced by medical reasons not to have progeny) are not and they may have a long-term impact on population growth. Thus we see the importance of studying population models that take into consideration non-reproductive groups.

It is intuitively obvious that in a population closed to migration, if the segregation rate into non-reproductive groups is large enough, the population will decline and, eventually, extinguish itself. In contrast, in a population with a steady immigration of reproductive individuals, this is not the case: even if every individual in the population were segregated into non-reproductive groups, the external supply of new individuals will be able to sustain a population, albeit an entire non-reproductive one.

Similar results can be expected for the fate of an infectious disease invading a population in which some proportion of infected individuals is isolated from all (infectious) contact with others for the duration of the infectious period. An example of such situation comes from sexually transmitted diseases when one considers not the reproduction of individuals *per se* but rather the reproduction of

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the disease, that is its transmission from infected to susceptible. A significant number of individuals infected with herpes simplex type B, for example, refrain from sexual activity during outbreaks when they are contagious. If a large enough proportion of them did that, the disease would eventually disappear. This, and other situations such as the impact of quarantine of infected individuals with SARS, will be analyzed in an upcoming paper.

2 One-Sex, Constant-Fertility and Mortality Models

The simplest unstructured demographic models prescribe a constant mortality rate per individual per unit of time, μ , and either a constant *total* birth rate per unit of time, Λ , or a constant *per capita* birth rate per unit of time, β .

The corresponding ODE models are, respectively,

$$P'(t) = \Lambda - \mu P(t) \quad \text{and} \quad P'(t) = \beta P(t) - \mu P(t), \quad (1)$$

with solutions

$$P(t) = P(0) e^{-\mu t} + \frac{\Lambda}{\mu} (1 - e^{-\mu t}) \quad \text{and} \quad P(t) = P(0) e^{(\beta - \mu)t}. \quad (2)$$

Both models have monotone behavior for $P(0) > 0$ —the former asymptotically converging to $\frac{\Lambda}{\mu}$, the latter growing (or decreasing) exponentially at Malthusian rate $r = \beta - \mu$.

These are usually too simplistic for long-term demographic projections since many factors that impact the evolution of the size of human populations are not taken into account.

If we assume there is a uniform probability per unit of time, $\nu \geq 0$, that a randomly selected individual will not reproduce because of any of the reasons mentioned in the introduction (or for any other reason), then we can modify the unstructured models (1) as follows to reflect this particular behavior: let $R(t)$ and $N(t)$ denote, respectively, the size of the reproductive and non-reproductive populations. Then, the dynamics of the reproductive group is described by one of the following equations,

$$R'(t) = \Lambda - \mu R(t) - \nu R(t) \quad \text{and} \quad R'(t) = \beta R(t) - \mu R(t) - \nu R(t), \quad (3)$$

while that of the non-reproductive group is described by

$$N'(t) = -\mu N(t) + \nu R(t). \quad (4)$$

The respective solutions for the reproductive group are

$$R(t) = R(0) e^{-(\mu + \nu)t} + \frac{\Lambda}{\mu + \nu} (1 - e^{-(\mu + \nu)t}) \quad \text{and} \quad R(t) = R(0) e^{(\beta - \mu - \nu)t}, \quad (5)$$

while those for the non-reproductive one are, respectively,

$$N(t) = \frac{\Lambda \nu}{\mu(\mu + \nu)} + \left[P(0) - \frac{\Lambda}{\mu} \right] e^{-\mu t} + \left[\frac{\Lambda}{\mu + \nu} - R(0) \right] e^{-(\mu + \nu)t}, \quad (6)$$

$$\begin{cases} N(t) = \left[N(0) - \frac{\nu R(0)}{\beta - \nu} \right] e^{-\mu t} + \frac{\nu R(0)}{\beta - \nu} e^{(\beta - \mu - \nu)t} & (\text{if } \beta \neq \nu), \\ N(t) = [N(0) + R(0)\nu t] e^{-\mu t} & (\text{if } \beta = \nu). \end{cases} \quad (7)$$

Of course, $\nu = N(0) = 0$ in (3)-(7) imply $R(t) \equiv P(t)$ and $N(t) \equiv 0$, whence (3) is just (1) and (5) is just (2).

Next note that, summing (5) with (6) and (7), we obtain explicit formulas for the total population size, $P = R + N$, for each model:

$$P(t) = \frac{\Lambda}{\mu} + \left[P(0) - \frac{\Lambda}{\mu} \right] e^{-\mu t}, \quad (8)$$

$$\begin{cases} P(t) = \left[N(0) - \frac{\nu R(0)}{\beta - \nu} \right] e^{-\mu t} + \frac{\beta R(0)}{\beta - \nu} e^{(\beta - \mu - \nu)t} & (\text{if } \beta \neq \nu), \\ P(t) = [P(0) + R(0)\nu t] e^{-\mu t} & (\text{if } \beta = \nu). \end{cases} \quad (9)$$

Comparing (2) and (8), we can readily see that in the first model the effect of the non-reproductive group on total population growth is nil. Also, from (6) and (8), we see that the proportion of non-reproductive individuals asymptotically approaches the value $\nu/(\mu + \nu)$.

On the other hand, concerning the second unstructured model in (1), we see from (9) that when $\beta > \mu$ and $\nu \neq \beta$, there is a threshold value for ν , $\nu_0 = \beta - \mu$, that if surpassed, leads to ultimate population extinction. Moreover, combining (7) and (9), we see that the proportion of non-reproductive individuals in the total population asymptotically approaches 1 if $\beta \leq \nu$, and it approaches the ratio ν/β if $\beta > \nu$. The latter is possible for exponentially decreasing populations ($\beta < \mu + \nu$), bounded populations ($\beta = \mu + \nu$), or exponentially increasing ones ($\beta > \mu + \nu$).

We also note that in the case $0 \leq \beta - \mu = \nu_0 < \nu < \beta$, if $N(0) = 0$, we can write

$$\begin{cases} P(t) = R(0) \left[\frac{\beta}{\beta - \nu} e^{(\beta - \mu - \nu)t} - \frac{\nu}{\beta - \nu} e^{-\mu t} \right], \\ P'(t) = R(0) \left[\frac{\beta(\beta - \mu - \nu)}{\beta - \nu} e^{(\beta - \mu - \nu)t} + \frac{\mu\nu}{\beta - \nu} e^{-\mu t} \right], \end{cases} \quad (10)$$

and thus we see that the population increases for $t \leq \frac{\ln\left(\frac{\mu\nu}{\beta(\mu + \nu - \beta)}\right)}{\beta - \nu}$ and then it decreases exponentially to 0. Similarly, for $\beta = \nu > \mu$, P increases for $t \leq \frac{\nu - \mu}{\mu\nu}$, and then it decreases exponentially to 0. If $\nu > \beta$, then P decreases exponentially to 0.

Example 1 As examples based on real-life data, we may point out the vital rates for Japan in 2001 were, $\beta = 0.0093$ and $\mu = 0.0077$, [4], that result in a threshold value $\nu_0 = 0.0016$ for this last model. If there was no migration, a segregation rate from the reproductive population into the non-reproductive one exceeding just 0.16% per year would lead to population extinction.

In the city of Buenos Aires, Argentina, for that same year the values were $\beta = 0.0139$ and $\mu = 0.0111$, [1], that result in a threshold value $\nu_0 = 0.0028$ for this last model. We show this last case in Figure 1. This threshold should be interpreted with caution as these values were estimated for the whole population, without consideration of its age-distribution. ■

3 One-Sex Logistic Models

A better model for long-term projections than either model (1) is that of logistic growth,

$$P' = r \left(1 - \frac{P}{K} \right) P, \quad (11)$$

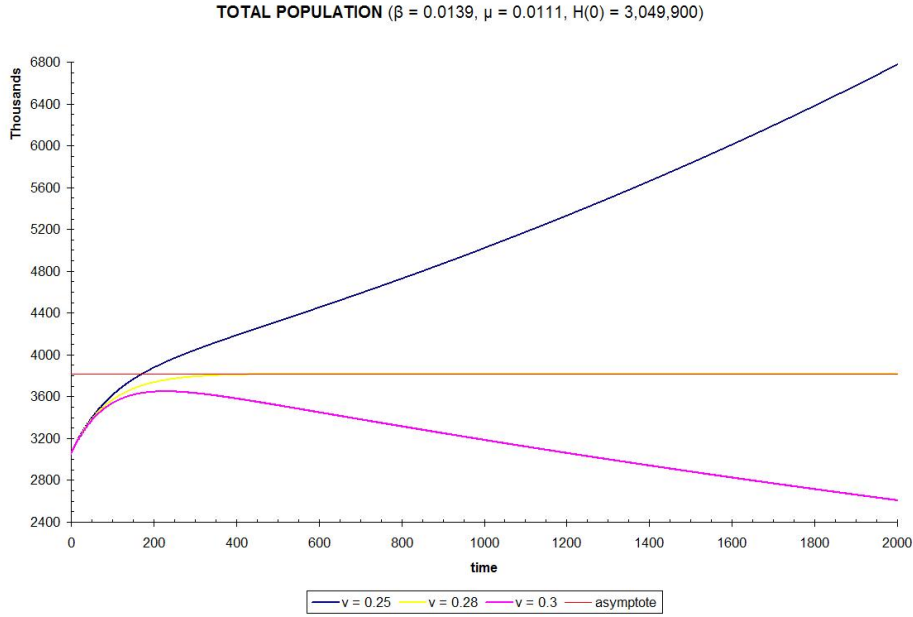


Figure 1: Example from Buenos Aires, Argentina, year 2001.

where K is the *carrying capacity of the environment* and r is the Malthusian rate, with solution

$$P(t) = \frac{K}{1 + \left[\frac{K}{P(0)} - 1\right] e^{-rt}},$$

asymptotically approaching the value K , for any non-zero initial value $P(0)$.

We can also modify the logistic model (11) to include segregation into a non-reproductive class as follows:

$$\begin{cases} R' &= (\beta - \mu) \left(1 - \frac{P}{K}\right) R - \nu R, \\ N' &= -\mu N - (\beta - \mu) \frac{P}{K} N + \nu R, \end{cases} \quad (12)$$

where the total population is then modeled by

$$P' = (\beta - \mu) \left(1 - \frac{P}{K}\right) P - \nu N.$$

Unfortunately, there is no explicit form of the solution of (12) for $\nu > 0$. However, the first equation in (12) exhibits a threshold phenomenon that separates population persistence from population extinction. Clearly, if $\nu > \beta - \mu > 0$, then $R(0) e^{-(\mu + \nu - \beta)t}$ is a super-solution and thus R must asymptotically approach zero. On the other hand, if $\nu < \beta - \mu$, then $R' = (\beta - \mu - \nu) R - (\beta - \mu) \frac{P}{K} R$ is positive as soon as $P < K \left(1 - \frac{\nu}{\beta - \mu}\right)$, giving a positive lower limit for P . In this sense we see that the logistic model exhibits the same threshold behavior as Malthus's exponential growth model: when $\beta > \mu$, they both represent declining populations that go asymptotically extinct if $\nu > \beta - \mu$,

and they both have populations that persist for all time if $\nu < \beta - \mu$ —the former asymptotically approaching a finite positive limit, the latter growing exponentially at rate $r = \beta - \mu - \nu$.

We present in Figure 2 below the size of the reproductive and the total population obtained from numerical solutions of (12) using the initial size, mortality and fertility rates from the actual US population in the year 2000: $R(0) = P(0) = 281$ million, $N(0) = 0$, carrying capacity $K = 300$ million (chosen just for reference), mortality rate $\mu = 0.01303$, *per capita* birth rate $\beta = 0.01442$, and several values of the segregation rate ν . We show the behavior of the logistic model in the absence of the non-reproductive group ($\nu = 0$), as well as the reproductive group size and total population size when the segregation rate is $\nu = 0.000125, 0.00025, 0.0005, 0.001$, and 0.002 . The proportion of heterosexual individuals in the total population asymptotically stabilizes to lower values with increasing ν , at approximately 99% for $\nu = 0.000125$, 98% for $\nu = 0.00025$, 97% for $\nu = 0.0005$, and 93% for $\nu = 0.001$. Of course, for $\nu = 0.002 > \beta - \mu$, the entire population asymptotically declines to extinction.

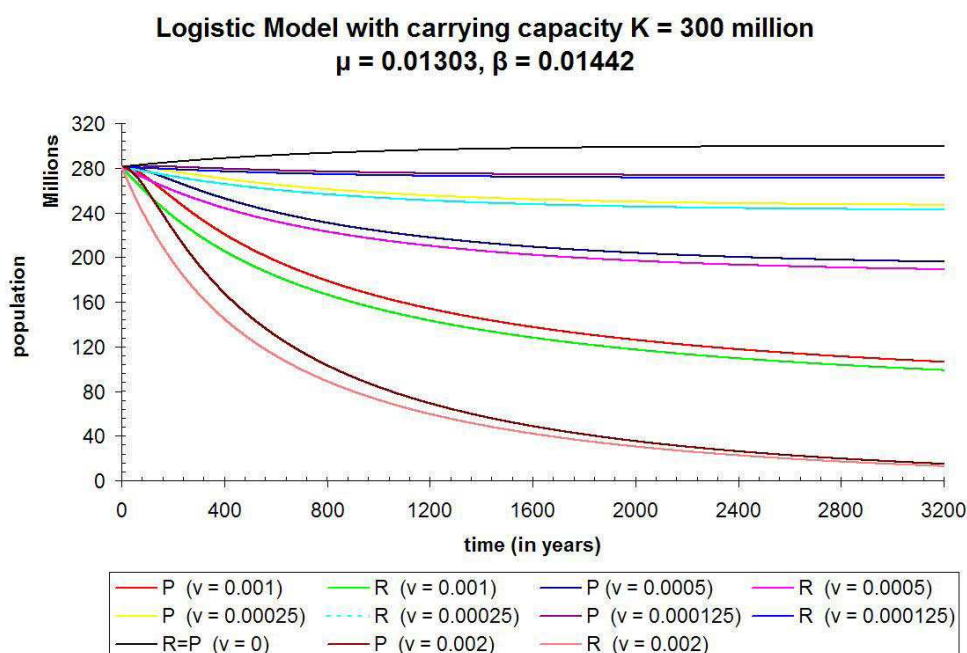


Figure 2: Example of the impact of segregation in logistic growth.

4 Two-Sex Models

When trying to model sexual reproduction, a simple model due to Kendall [2] is described in terms of the numbers of uncoupled females, F , uncoupled males, M , and reproductive couples, C . Let now μ_f and μ_m be, respectively, the per capita mortality rates of females and males, β represent the *per couple* birth rate per unit of time, γ represent the proportion of newborn males among all newborn (e.g. $\gamma = \frac{2,076,969}{4,058,815} = 0.511718$ for the U.S.A. in the year 2000), δ be the per couple divorce rate per unit of time, and \mathcal{M} represent the coupling function giving the number of new reproductive pairings per unit of time when F single (i.e. uncoupled) female and M single male individuals are present in the population. Then, the ODE model is given by

$$\begin{cases} F' &= -\mu_f F + (1 - \gamma)\beta C - \mathcal{M}(F, M) + \delta C, \\ M' &= -\mu_m M + \gamma\beta C - \mathcal{M}(F, M) + \delta C, \\ C' &= -(\delta + \mu_f + \mu_m)C + \mathcal{M}(F, M). \end{cases} \quad (13)$$

We let now ν_f and ν_m be, respectively, the probability per unit of time that a single (uncoupled) female or male newborn will never reproduce (for any reason). Then, we modify (13) as follows:

$$\begin{cases} F' &= -\mu_f F - \nu_f F + (1 - \gamma)\beta C - \mathcal{M}(F, M) + \delta C, \\ M' &= -\mu_m M - \nu_m M + \gamma\beta C - \mathcal{M}(F, M) + \delta C, \\ C' &= -(\delta + \mu_f + \mu_m)C + \mathcal{M}(F, M). \end{cases} \quad (14)$$

If we let $\varphi(t)$ and $\chi(t)$ represent, respectively, the numbers of non-reproductive females and males in the population at time t , then we can model their dynamics as follows:

$$\begin{cases} \varphi' &= -\mu_f \varphi + \nu_f F, \\ \chi' &= -\mu_m \chi + \nu_m M. \end{cases} \quad (15)$$

There is no explicit solution to the system (14)-(15) in general. However, if we make the assumptions $\gamma = \frac{1}{2}$, $\mu_f = \mu_m$, $\nu_f = \nu_m$, $\mathcal{M} = 2\rho \frac{FM}{F+M}$, and $F(0) = M(0)$, it follows that $F(t) = M(t)$ for all $t \geq 0$ and system (14) reduces to

$$\begin{cases} F' &= -(\mu + \nu)F + \frac{\beta}{2}C - \rho F + \delta C, \\ C' &= -(\delta + 2\mu)C + \rho F, \end{cases} \quad (16)$$

while assuming in addition that $\varphi(0) = \chi(0)$ implies $\varphi(t) \equiv \chi(t)$. It is possible to determine conditions on the demographic rates that will make F a multiple of C and further reduce system (16) to a single ODE that can be explicitly solved. Letting $F = \alpha C$ in (16), we see that

$$\alpha C' = -(\mu + \nu)\alpha C + \frac{\beta}{2}C - \rho\alpha C + \delta C = -(\delta + 2\mu)\alpha C + \rho\alpha^2 C,$$

whereby

$$\rho\alpha^2 + (\rho + \nu - \mu - \delta)\alpha - \left(\frac{\beta}{2} + \delta\right) = 0. \quad (17)$$

Since the discriminant of this equation is always positive, it follows that the equation has two real roots, one positive and one negative. The positive one,

$$\alpha = \frac{\mu + \delta - \rho - \nu + \sqrt{(\rho + \nu - \mu - \delta)^2 + 2\rho(\beta + 2\delta)}}{2\rho}, \quad (18)$$

gives again the multiplier that relates F and C . With this value of α we now solve the second equation in (16) for C , namely $C' = r_\nu C$:

$$C(t) = C(0) e^{r_\nu t},$$

with Malthusian rate $r_\nu = \rho\alpha - \delta - 2\mu$, dependent upon ν through α . Finally, we obtain for the singles

$$F(t) = M(t) = \alpha C(0) e^{r_\nu t} = F(0) e^{r_\nu t},$$

and, solving (15) using this expression for F , we obtain for the non-reproductive group

$$\begin{cases} \varphi(t) = \left[\varphi(0) - \frac{\alpha\nu C(0)}{\mu + r_\nu} \right] e^{-\mu t} + \frac{\alpha\nu C(0)}{\mu + r_\nu} e^{r_\nu t} & (\text{if } \rho\alpha \neq \mu + \delta), \\ \varphi(t) = [\varphi(0) + \alpha\nu C(0) t] e^{-\mu t} & (\text{if } \rho\alpha = \mu + \delta). \end{cases} \quad (19)$$

The total population size, $P(t) = 2[F(t) + \varphi(t) + C(t)]$, is now

$$\begin{cases} P(t) = 2 \left[1 + \alpha + \frac{\alpha\nu}{\mu + r_\nu} \right] C(0) e^{r_\nu t} + 2 \left[\varphi(0) - \frac{\nu F(0)}{\mu + r_\nu} \right] e^{-\mu t} & (\text{if } \rho\alpha \neq \mu + \delta), \\ P(t) = 2[1 + \alpha] C(0) e^{r_\nu t} + 2[\varphi(0) + \alpha\nu C(0) t] e^{-\mu t} & (\text{if } \rho\alpha = \mu + \delta). \end{cases} \quad (20)$$

It follows from (19) and (20) that the proportion of non-reproductive individuals in the population asymptotically approaches 1 if $\rho\alpha \leq \mu + \delta$, and it approaches the ratio $\alpha\nu/[\alpha\nu + (\alpha + 1)(\rho\alpha - \delta - \mu)]$ if $\rho\alpha > \mu + \delta$.

We note from (18) that

$$r_\nu = \rho\alpha - \delta - 2\mu = \frac{-(3\mu + \rho + \nu + \delta) + \sqrt{(\rho + \nu - \mu - \delta)^2 + 2\rho(\beta + 2\delta)}}{2}$$

vanishes if, and only if, $(\rho + \nu - \mu - \delta)^2 + 2\rho(\beta + 2\delta) = (3\mu + \rho + \nu + \delta)^2$ or, equivalently,

$$\nu = \frac{\rho\beta - 2\mu(2\mu + 2\rho + \delta)}{2(\delta + 2\mu)}. \quad (21)$$

If we assume that *in the absence of non-reproductive groups* the population increases exponentially at the rate

$$r_0 = \frac{-(3\mu + \rho + \delta) + \sqrt{(\rho - \mu - \delta)^2 + 2\rho(\beta + 2\delta)}}{2} > 0,$$

it follows that $(\rho - \mu - \delta)^2 + 2\rho(\beta + 2\delta) > (3\mu + \rho + \delta)^2$, that is

$$\rho\beta - 2\mu(2\mu + 2\rho + \delta) > 0.$$

Thus, we see from (21) a threshold effect whereby a rate of passage into non-reproductive groups

$$\nu_f = \nu_m = \nu = \frac{\rho\beta - 2\mu(2\mu + 2\rho + \delta)}{2(\delta + 2\mu)} > 0$$

would stop the exponential growth and result in a stagnant population. Higher values of ν would asymptotically lead to population extinction.

Next, let us assume $M(t) = F(t)$ for all time, harmonic mean marriage function, and look for exponential solutions with $M = F = \alpha C$, we see from (14) that

$$-\alpha(\mu_f + \nu_f) + \beta(1 - \gamma) = -\alpha(\mu_m + \nu_m) + \beta\gamma. \quad (22)$$

It follows from (22) that either

$$\mu_m + \nu_m = \mu_f + \nu_f \quad \text{and, necessarily,} \quad \gamma = \frac{1}{2},$$

or

$$\mu_m + \nu_m \neq \mu_f + \nu_f \text{ and, necessarily, } \alpha = \frac{\beta(2\gamma - 1)}{(\mu_m - \mu_f) + (\nu_m - \nu_f)}. \quad (23)$$

The first case leads, of course, to (16). In the second case, however, α also needs to satisfy the relation $\rho\alpha^2 + (\rho - \mu_f + \nu_m - \delta)\alpha - (\beta\gamma + \delta) = 0$, corresponding to (17) for this case, giving the following necessary condition for $F(t)$ to be identical with $M(t)$ for all time in the case of unequal vital rates for the two genders:

$$\frac{\mu_f + \delta - \rho - \nu_m + \sqrt{(\rho - \mu_f + \nu_m - \delta)^2 + 4\rho(\beta\gamma + \delta)}}{2\rho} = \frac{\beta(2\gamma - 1)}{(\mu_m - \mu_f) + (\nu_m - \nu_f)}. \quad (24)$$

Of course, $\nu_f = \nu_m = 0$ together with $\phi(0) = \chi(0) = 0$ reduce (14) to (13), in which case (18) simplifies to

$$\alpha = \frac{\mu + \delta - \rho + \sqrt{(\rho - \mu - \delta)^2 + 2\rho(\beta + 2\delta)}}{2\rho}, \quad (25)$$

giving the multiplier that relates F and C .

The total population in this case is given by $P = 2F + 2C = 2(\alpha + 1)C$, that is $C = \frac{1}{2(\alpha+1)}P$. Therefore, using (16), we see that it satisfies $P' = rP$, with

$$r = \frac{\beta - 2\mu(\alpha + 2)}{2(\alpha + 1)}. \quad (26)$$

We note from (25) and (26) that

$$\begin{aligned} r &= \rho\alpha - \delta - 2\mu > 0 \\ &\iff (\rho - \mu - \delta)^2 + 2\rho(\beta + 2\delta) > (3\mu + \rho + \delta)^2 \\ &\iff 2\mu(2\mu + \delta) < \rho(\beta - 4\mu) \\ &\iff \beta > 4\mu \text{ and } \rho > 2\mu \frac{2\mu + \delta}{\beta - 4\mu}. \end{aligned}$$

This says that the population will increase provided both conditions $\beta > 4\mu$ and $\rho > 2\mu \frac{2\mu + \delta}{\beta - 4\mu}$ are satisfied. The first one means that the fertility rate must exceed the total mortality rate, while the second one means that the marriage rate is sufficiently large to produce enough newborn.

It is easy to verify that $\lim_{\rho \rightarrow \infty} \alpha = 0$, as it should necessarily be since in the limiting case $\rho \rightarrow \infty$ all individuals are married, making $F = 0C$. Also, $\lim_{\rho \rightarrow \infty} r = \frac{1}{2}(\beta - 4\mu)$, which reconciles the two-sex model with the one-sex model, since (16) has, in this limiting case, a fertility rate $\beta/2$ (i.e. the average couple produces that many female newborn and an equal number of male newborn that instantaneously become $\beta/2$ new couples), while the mortality rate for couples is 2μ , independently of the value of δ , as divorce has no impact in the number of couples when the marriage rate is infinitely large.

We see from (23) that in the absence of non-reproductive groups, since α is greater than 0, $\gamma > \frac{1}{2}$ implies $\mu_m > \mu_f$. This is actually the case for human populations.

Example 2. As indicated earlier, for the U.S.A. in the year 2000, $\gamma = 0.5117$ and the mortality rates, computed as reciprocals of the life expectancy at birth, were $\mu_f = 0.01258$ and $\mu_m = 0.01350$, [3]. For that same year, $\beta = 0.07338$ (approximately 4.059 million births from 55.311 million couples)

making $\alpha = 1.8664$. However, α also needs to satisfy the relation $\rho\alpha^2 + (\rho - \mu_f - \delta)\alpha - (\beta\gamma + \delta) = 0$, corresponding to (25) for this case, giving the following necessary condition for $F(t)$ to be identical with $M(t)$ for all time in the case of unequal vital rates for the two genders:

$$\alpha = \frac{\mu_f + \delta - \rho + \sqrt{(\rho - \mu_f - \delta)^2 + 4\rho(\beta\gamma + \delta)}}{2\rho} = \frac{\beta(1 - 2\gamma)}{\mu_f - \mu_m}. \quad (27)$$

In the year 2000 there were approximately 2,376,000 marriages from approximately 88.057 million unmarried females and 82.743 million unmarried males, resulting in $\rho = 0.027849$. It then follows from (27) that δ must satisfy the relation

$$\delta = \frac{\zeta^2 - 4\rho\beta\gamma - (\rho - \mu_f)^2}{2(\rho + \mu_f + \zeta)},$$

where we have set

$$\zeta = \frac{2\rho\beta(1 - 2\gamma)}{\mu_f - \mu_m} + \rho - \mu_f.$$

Finally we see that, with $\zeta = 0.1192$, $\delta = 0.03069$, and initial conditions

$$F(0) = M(0) = 1.8664 C(0), \quad (28)$$

the system (13) would have the solution $F(t) = M(t) = 1.8664 C(0) e^{rt}$, with Malthusian growth rate $r = \rho\alpha - \delta - \mu_f - \mu_m = -0.004795$. This “model” population decreases at a rate of approximately 0.5% per year. Note that, aside from the fact that the total numbers of unmarried females, unmarried males, and married couples in the U.S. in the year 2000 do not satisfy (28), the actual divorce rate for the year 2000 computed as the ratio of 0.9572 million divorces (excluding those from CA, IN, LA, and OK, where no divorce statistics are recorded) to an estimated 45.784 million married couples outside those four States —number obtained as the proportion of the U.S. population living outside those four States multiplied by $C(0) = 55.311$ million couples in the U.S.— is $\delta = 0.02091$, approximately 33% lower than 0.03069. For this reason, it is to be expected that the Malthusian growth rate of the “model” population should be lower than that of the actual U.S. population found from the one-sex model (1), $r = \beta - \mu = 0.01442 - 0.01303 = 0.00139$, or approximately 0.14% per year.

These calculations underscore the importance of distinguishing among the two sexes when trying to determine the Malthusian rate for a human population, since the assumption that their numbers are identical may result in errors in the Malthusian that could even make it uncertain whether the population is increasing or decreasing. We shall redo these calculations in Example 3, without assuming that the numbers of females and males are identical for all time.

It is worth noting that, using $\delta = 0.02091$ in (27), we obtain $\alpha = 1.55365$ and, using then the right-hand side of (27), we find $\mu_m = 0.013685$ —just 1.4% larger than the actual per capita male mortality rate for the U.S. in the year 2000. This observation stresses the sensitivity of these calculations to changes in some of the parameters involved.

Finally we remark that, using the actual vital rates the right-hand side of (27) gives $\alpha = 1.8664$ and thus $\delta = \alpha\rho - r - \mu_f - \mu_m = 1.8664\rho - 0.02747$. Substituting this relation in the first equality of (27) we see that there is no solution for ρ —positive or negative. This summarizes the fact that the actual vital rates for the U.S. in the year 2000 do not correspond to those of a two-sex population with equal numbers of females and males for *any* marriage and divorce rates. ■

In general, if we assume marriages take place according to the harmonic mean and still look for exponential solutions with a common Malthusian rate, we can set $F = \alpha C$, $M = \xi C$, and letting

$\sigma = \delta + \mu_f + \mu_m$ be the separation rate per couple, we immediately obtain from (13)

$$\begin{cases} -\alpha\mu_f + \beta(1 - \gamma) - 2\rho\frac{\alpha\xi}{\alpha + \xi} + \delta = -\alpha\sigma + 2\rho\frac{\alpha^2\xi}{\alpha + \xi}, \\ -\xi\mu_m + \beta\gamma - 2\rho\frac{\alpha\xi}{\alpha + \xi} + \delta = -\xi\sigma + 2\rho\frac{\alpha\xi^2}{\alpha + \xi}, \end{cases} \quad (29)$$

and, subtracting these relations side-by-side,

$$(\xi\mu_m - \alpha\mu_f) + \beta(1 - 2\gamma) = (\alpha - \xi) \left[2\rho\frac{\alpha\xi}{\alpha + \xi} - \sigma \right],$$

whereby, if $\alpha \neq \xi$,

$$2\rho\frac{\alpha\xi}{\alpha + \xi} - \sigma = \frac{\beta(1 - 2\gamma) + (\xi\mu_m - \alpha\mu_f)}{\alpha - \xi}. \quad (30)$$

Combining (13) with (30), we see that $C(t) = C(0) e^{rt}$, where the Malthusian rate is

$$r = \frac{\beta(1 - 2\gamma) + (\xi\mu_m - \alpha\mu_f)}{\alpha - \xi}. \quad (31)$$

Next, multiplying both sides of the equations by $(\alpha + \xi)$, we rewrite system (29) in the following form:

$$\begin{cases} \xi [\beta(1 - \gamma) + \delta + (\sigma - \mu_f - 2\rho - 2\rho\alpha)\alpha] = [(\mu_f - \sigma)\alpha - \beta(1 - \gamma) - \delta]\alpha \\ \alpha [\beta\gamma + \delta + (\sigma - \mu_m - 2\rho - 2\rho\xi)\alpha] = [(\mu_m - \sigma)\xi - \beta\gamma - \delta]\xi, \end{cases}$$

that can be readily rearranged as

$$\begin{cases} \xi = \frac{[(\mu_f - \sigma)\alpha - \beta(1 - \gamma) - \delta]\alpha}{\beta(1 - \gamma) + \delta + (\sigma - \mu_f - 2\rho - 2\rho\alpha)\alpha} \\ (-2\rho\alpha + \sigma - \mu_m)\xi^2 + [(\sigma - \mu_m - 2\rho)\alpha + \beta\gamma + \delta]\xi + (\beta\gamma + \delta)\alpha = 0. \end{cases} \quad (32)$$

If we substitute now the first equation in (32) into the second, we obtain a quintic for α , whose positive real roots (if any), together with (32.2), lead to the sought exponential solution.

Example 3. Using the data for the population of the U.S. in the year 2000 as given in Example 2, we have $\sigma = 0.04699$, $\xi \approx -\frac{-0.03441\alpha^2 - 0.05674\alpha}{-0.055698\alpha^2 - 0.021288\alpha + 0.05674}$ and

$$(-0.055698\alpha + 0.03349)\xi^2 + (-0.022208\alpha + 0.058458546)\xi + 0.058458546\alpha = 0,$$

i.e.

$$\xi = \frac{(0.022208\alpha - 0.058459) \pm \sqrt{0.013517\alpha^2 - 0.010428\alpha + 0.0034174}}{0.06698 - 0.11140\alpha}$$

There is a unique real positive solution, $(\alpha, \xi) = (1.536, 1.568)$. Using (31), it gives a growth rate $r = -0.00400$. Note that the Malthusian rate computed in Example 2 assuming equal numbers of females and males is 20% smaller than the value we just found, stressing again the quantitative importance of keeping the sexes separate in the model. ■

As a final remark, we shall explain why the Malthusian rate computed from the two-sex model is so much smaller than that computed from the one-sex model —indeed the former is negative

while the latter is positive. In the one-sex model the whole population is reproductive and the only two parameters that may affect its long term growth are the *vital rates*, fertility and mortality rates. In the two-sex model under consideration, only couples are reproductive and, therefore, if a two-sex population is to grow it is not only necessary that fertility exceed total mortality, but also that there be enough marriages —as indicated in the argument just following (26). The necessary and sufficient conditions derived for the case $\gamma = \frac{1}{2}$ and gender-independent vital rates are $\beta > 4\mu$ and $\rho > 2\mu \frac{2\mu+\delta}{\beta-4\mu}$. The analogous conditions when the vital rates differ among the sexes are $\beta > 2(\mu_f + \mu_m)$ and $\rho > \frac{\sigma(\mu_f + \mu_m)}{\beta - 2(\mu_f + \mu_m)}$. For the population of the U.S. in the year 2000 the first one is clearly satisfied and the second one is clearly not. This is the real reason why the Malthusian rate for the two-sex model population that results from using the actual data for the year 2000 is negative. Part of the problem can be solved by considering births not only from couples but from single females as well. Thus a need for even more detailed models arises and some already exist and others will be developed elsewhere.

5 Conclusions

We considered several simple population models and introduced in them non-reproductive classes that may represent, for example, lifelong childless homosexuals, individuals incarcerated until the end of their reproductive years, or others. We studied here the impact that such groups can have on the growth of the total population.

For the simplest model, a “learning curve” corresponding to a population with a constant external supply of new individuals, we showed that the segregation into a non-reproductive group does not alter the growth dynamics of the total population but it rather structures it into a reproductive class consisting of a proportion $\frac{\mu}{\mu+\nu}$ of the total, and a non-reproductive class consisting of a proportion $\frac{\nu}{\mu+\nu}$ of the total.

For the next simplest model, Malthus’s exponential model, we showed that the segregation into a non-reproductive group can indeed alter the population trend and make an exponentially increasing population stagnate or decline (depending on how the *per capita* birth rate compares with the sum of the mortality and segregation rates), and that the proportion of non-reproductive individuals in the total population will either approach 1 or the ratio $\frac{\nu}{\beta}$ (depending on whether $\beta \leq \nu$ or $\beta > \nu$).

For the logistic model we showed similar results, namely that the segregation into a non-reproductive group can indeed alter the population trend and make an increasing logistic population stagnate or decline (depending on how the *per capita* birth rate compares with the sum of the mortality and segregation rates), but we were not able to derive this time an explicit expression for the asymptotic proportion of non-reproductive individuals in the population.

Some important behavioral considerations of human populations are neglected in the models in this paper, such as the role and impact of bisexual individuals on population growth, births from homosexual females, return of individuals from homosexual groups to heterosexual ones, differences in contact rates among homosexuals forming couples and those who do not, and others. These issues will be addressed in an upcoming paper.

Finally, the importance of segregation in population control discussed in this paper, will be extended to the context of disease transmission and control for sexually transmitted diseases. The impact of isolating infectious individuals from sexual activity on the decline of disease prevalence will be analyzed elsewhere. There is an obvious analogy between the impact that segregation of reproductive individuals into non-reproductive groups has on population growth, and the impact that the segregation of individuals infected with a sexually transmitted disease into non-sexually active groups will have on the growth of the infected class.

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