A Numerical Method for Nonlinear Age-Structured Population Models with Finite Maximum Age

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Abstract

We propose a new numerical method for the approximation of solutions to a non-autonomous form of the classical Gurtin-Mac Camy population model with a mortality rate that is the sum of an intrinsic age-dependent rate that becomes unbounded as the age approaches its maximum value, plus a non-local, non-autonomous, bounded rate that depends on some weighted population size. We prove that our new quadrature based method converges to second-order and we show the results of several numerical simulations.

Key words: age-structured population, approximation, numerical methods

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1 Introduction

The mathematical theory of age-structured populations is quite well developed [8,9]. They play a major role in ecological modeling, as well as in demography, epidemiology, cancer modeling, and other fields. Many models simplify the mathematical analysis by assuming that the mortality rate—as well as all other modeling parameters—is bounded, automatically leading to the possibility of immortality. Some more realistic models impose a maximum age that cannot be reached and must require that the mortality rate become unbounded at that age. Lotka [] and McKendrick [] are credited with the first age-structured model, a linear one that supports exponential solutions, just as the unstructured Malthus [] model. A nonlinear form of that model was first proposed by Morton Gurtin and Richard Mac Camy [] by making the fertility and mortality rates dependent on the total population size, that is the integral of the age density.

Numerical methods to approximate the solution of such population models have been proposed and analyzed during the past twenty-five years. For an excellent review of these methods see [1]. Two of the main objectives driving the need for numerical methods are, first, the need to make projections about population growth for the future, usually for periods of 10-50 years. Secondly, there is the theoretical interest in long-term simulations for the purpose of analyzing trends under different scenarios. This is an important aspect of population models in theoretical biology. Some examples of numerical studies of structured population dynamics are found, for example, in [2,**?**,4,5,7,13].

For the use of real life data we have to limit ourselves to low regularity in the data, thus limiting the practicality of the numerical methods to second-order requiring just one continuous derivative for the coefficients. For long-term simulations higher-order methods would be more desirable and, therefore, second-order methods are a good compromise between performance and regularity demands. The first to point out that standard numerical methods based on uniform meshes degenerate and do not converge near the age of unbounded mortality were Iannelli and Milner [10]. Kim and Kwon [11] introduced a numerical method that reaches its optimal order of convergence when the tail of the mortality function has some specific analytic form. Even though the partial differential equations in the model can be readily integrated along the characteristics as ordinary differential equations—thus providing natural numerical methods that consist of approximating that integral representation—very few numerical methods are based on that explicit analytic representation.

In order to provide more real life applicability, we consider a nonlinear population model with finite maximum age. The new method we propose and analyze is based on quadratures of the integrals that appear in the explicit representation of solutions obtained by integration along characteristics. It converges at its optimal order—second—for more general tails that include some for which other methods in the literature fail. In particular, our method is applicable to families of mortality functions that include those most often used in population modeling.

The paper is organized as follows: the second section is devoted to a detailed description of the model. In Section 4 we present the convergence analysis of the numerical method introduced in Section 3. Finally, the fifth Section is devoted to numerical simulations and conclusions.

2 The Model

We consider the problem of modeling the evolution of the age density of a population given by the following classical system:

$$u_t + u_a = -(m(a) + \mu(a, I_\mu(t), t))\, u, \quad 0 < a < a_\dagger,\ t > 0, \tag{2.1}$$

$$u(0,t) = \int_0^{a_\dagger} \alpha(a, I_\alpha(t), t)\, u(a,t)\, da\ , \quad t > 0, \tag{2.2}$$

$$u(a,0) = u_0(a), \quad 0 \le a \le a_\dagger, \tag{2.3}$$

$$I_s(t) = \int_0^{a_\dagger} \gamma_s(a)\, u(a,t)\, da, \quad t > 0, \quad s = \mu, \alpha, \tag{2.4}$$

where the independent variables a and t represents respectively age and time. The function $u(\cdot,t)$ is the age density of the population at time t and the vital functions are given by the age-specific mortality rate $(m(\cdot) + \mu(\cdot, I_\mu(t), t))$ and the age-specific fertility rate $(\alpha(\cdot, I_\alpha(t), t))$. We consider the mortality rate given in separable form consisting of two terms, an intrinsic mortality $m(a)$ —that is unbounded to take into account a maximum age $a_\dagger$— and a bounded mortality that, as the fertility rate, includes seasonality (through the dependency on the time) and resource competition (through the dependency on the non local functionals $I_s(t)$). Finally, $u_0(\cdot)$ denotes the initial age distribution. A derivation, analysis of well-posedness, and asymptotic behavior of solutions can be found, for example, in [8].

In the present paper, we introduce a second order numerical method to obtain the solution of this nonlinear model and we also carry out its convergence analysis. Throughout the paper we assume the following regularity conditions on the data functions and the solution of problem (2.1)-(2.4):

(H1) • $u \in \mathcal{C}^2([0, a_\dagger) \times [0,T])$, $u(a,t) \ge 0$, $a \in [0, a_\dagger]$, $t \ge 0$.
(H2) • $\gamma_\mu, \gamma_\alpha \in \mathcal{C}^2([0, a_\dagger])$.
(H3) • $m \in \mathcal{C}^2([0, a_\dagger))$, is nonnegative and $\displaystyle\int_a^{a_\dagger} m(\sigma)\, d\sigma = +\infty$.
(H4) • $\mu \in \mathcal{C}^2([0, a_\dagger] \times D_\mu \times [0,T])$, is nonnegative, where D_μ is a compact neighborhood of

$$\left\{ \int_0^{a_\dagger} \gamma_\mu(a)\, u(a,t)\, da,\ 0 \le t \le T \right\}.$$

(H5) • $\alpha \in \mathcal{C}^2([0, a_\dagger] \times D_\alpha \times [0,T])$, is nonnegative, where D_α is a compact neighborhood of

$$\left\{ \int_0^{a_\dagger} \gamma_\alpha(a)\, u(a,t)\, da,\ 0 \le t \le T \right\}.$$

3 The numerical method

The numerical method that we propose integrates the model along the characteristiclines $a - t = c$, c constant, where

$$\frac{d}{dt}u(t+c,t) = -(m(t+c) + \mu(t+c, I_\mu(t), t))\, u(t+c,t). \tag{3.1}$$

Therefore, the integral representation of the solutions of (3.1) along the characteristics is given by the following relation: for each $\bar{a}$, with $0 < \bar{a} < a_\dagger$, and $h > 0$ such that $\bar{a} + h < a_\dagger$,

$$u(a_0+h, t_0+h) = u(a_0,t_0)\, \exp\left(-\int_0^h \left[m(\bar{a}+\tau) + \mu(a_0+\tau, I_\mu(t_0+\tau), t_0+\tau) \right]\, d\tau \right). \tag{3.2}$$

The initial difficulty to produce a numerical method for a problem like (2.1)-(2.4) is that the intrinsic mortality is unbounded. Therefore, we consider an intermediate value $A^* \in (0, a_\dagger)$ such that m is bounded in $[0, A^*]$, and we know the function $f(a) = \int_{A^*}^a \mu(\sigma)\, d\sigma$, $A^* \le a \le a_\dagger$, and $f(a_\dagger) = +\infty$, [11,6]. Note that, in order to model the dynamics of a specific population, the parameters A^* and $a_\dagger$, and the function μ should be determined from the field data [6].

The numerical method we propose consists of the discretization of (3.2). First, we introduce an age grid on $[0, a_\dagger]$ but, taking into account the value A^* that we want to keep identified, we introduce a positive integer J^*, and we define the step size $h = A^*/J^*$. Then, the total number of grid points is given by $J = [a_\dagger/h]$, where $[\cdot]$ denotes the integer part and the nodes of the uniform partition of the interval $[0, a_\dagger]$ are $a_j = j\,h$, $0 \le j \le J$ (note that $a_{J^*} = A^*$ and $a_J \le a_\dagger$). We shall also use the notation $a_{j+\frac{1}{2}} = a_j + \frac{h}{2} = (j+\frac{1}{2})h$ to the denote the "midpoint nodes." We will integrate the problem in a fixed time interval $[0, T]$, so we define the discrete time levels, $t_n = n\,h$, $0 \le n \le N$, where $N = [T/h]$, as well as the intermediate time levels $t_{n+\frac{1}{2}} = t_n + \frac{h}{2} = (n+\frac{1}{2})h$.

The notation U_j^n will represent the numerical approximation to $u(a_j, t_n)$, $0 \le j \le J$, $0 \le n \le N$ (the subscript j refers to the age grid point a_j and the superscript n to the time level t_n). We also denote these approximations in vector form: $\mathbf{U}^n = [U_0^n, U_1^n, \ldots, U_J^n]$, $0 \le n \le N$.

Given approximations of the initial age density on the age grid,

$$\mathbf{U}^0 = [U_0^0, U_1^0, \ldots, U_J^0], \quad U_j^0 = u_0(a_j), \quad 0 \le j \le J,$$

(which are typically the grid restrictions of the initial density) the numerical method is defined by the following recursion that provides the numerical approximation at the time level $n+1$, ($\mathbf{U}^{n+1}$), from that at the time level n, ($\mathbf{U}^n$), $0 \le n \le N-1$:

$$U_{j+1}^{n+1} = U_j^n \, \exp\left(-h\,\left[m(a_{j+\frac{1}{2}}) + \mu(a_{j+\frac{1}{2}}, Q_h^*(\boldsymbol{\gamma}_\mu\,\mathbf{U}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}})\right]\right), \qquad 0 \le j \le J^* - 1, \tag{3.3}$$

$$U_{j+1}^{n+1} = U_j^n \, e^{[f(a_j)-f(a_{j+1})]} \, \exp\left(-h\,\mu\left(a_{j+\frac{1}{2}}, Q_h^*(\boldsymbol{\gamma}_\mu\,\mathbf{U}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}}\right)\right), \qquad J^* \le j \le J-1, \tag{3.4}$$

$$U_0^{n+1} = Q_h(\boldsymbol{\alpha}(\mathbf{U}^{n+1})\,\mathbf{U}^{n+1}), \tag{3.5}$$

$0 \le n \le N-1$, where $\boldsymbol{\alpha}(\mathbf{U}^n)_j = \alpha\left(a_j, Q_h(\boldsymbol{\gamma}_\alpha\,\mathbf{U}^n), t_n\right)$, $0 \le j \le J$, and the values at the half time-step are computed by means of

$$U_{j+1}^{n+\frac{1}{2}} = U_j^n \, \exp\left(-\frac{h}{2}\,\left[m\left(a_j\right) + \mu\left(a_j, Q_h(\boldsymbol{\gamma}_\mu\,\mathbf{U}^n), t_n\right)\right]\right), \quad 0 \le j \le J^* - 1, \tag{3.6}$$

$$U_{j+1}^{n+\frac{1}{2}} = U_j^n \, e^{[f(a_j)-f(a_{j+1/2})]} \, \exp\left(-\frac{h}{2}\,\mu\left(a_j, Q_h(\boldsymbol{\gamma}_\mu\,\mathbf{U}^n), t_n\right)\right), \qquad J^* \le j \le J-1. \tag{3.7}$$

The notations $Q_h(\mathbf{V})$ and $Q_h^*(\mathbf{V})$ represent, respectively, the second-order and the first-order composite open quadrature rules to approximate an integral over the interval $[0, a_\dagger]$ given by

$$Q_h(\mathbf{V}) = h\,V_1 + \sum_{j=1}^{J-1} \frac{h}{2}\left(V_j + V_{j+1}\right), \quad \mathbf{V} = (V_0, V_1, \ldots, V_J), \tag{3.8}$$

$$Q_h^*(\mathbf{V}) = \sum_{j=1}^{J} h\,V_j, \quad \mathbf{V} = (V_1, \ldots, V_J), \quad \text{the standard end-point rule.} \tag{3.9}$$

Furthermore, $\boldsymbol{\gamma}_s\,\mathbf{U}^n$, $0 \le n \le N$, $\boldsymbol{\gamma}_s\,\mathbf{U}^{n+\frac{1}{2}}$, $0 \le n \le N-1$, $s = \mu$, α, and $\boldsymbol{\alpha}(\mathbf{U}^n)\,\mathbf{U}^n$, $0 \le n \le N$, denote the componentwise product of the corresponding vectors. It is important to notice that the numerical method in (3.3)-(3.5) is explicit which represents a great advantage in its implementation as compared to an implicit formulation.

4 Convergence analysis

We begin the analysis of the numerical method that we have described in Section 3. Convergence will be obtained by means of consistency and nonlinear stability. First, we rewrite the numerical method into the discretization framework developed by López-Marcos *et al.* [12]. From now on, C will denote a positive constant which is independent of h, n ($0 \le n \le N$) and j ($0 \le j \le J$); C has possibly different values in different places.

We assume that the age discretization parameter h takes values in the set $H = \{A^*/J^*, \ J^* \in \mathbb{N}\}$ and $J = [a_\dagger/h]$. In addition, we set $N = [T/h]$. For each $h \in H$, we define the space

$$X_h = \left(\mathbb{R}^{J+1}\right)^{N+1},$$

where $\mathbb{R}^{J+1}$ is used to consider the approximations to the theoretical solution on the grid nodes, $0 \le n \le N$. We also introduce the space

$$Y_h = \mathbb{R}^{J+1} \times \mathbb{R}^N \times \left(\mathbb{R}^J\right)^N,$$

where we consider the residuals which arise from the initial approximations (first term in the product), from the approximation to the solution at the boundary node (second term), and from the approximations to the solution at the other grid nodes, for every time step (except the first one). We note that the spaces, X_h and Y_h, have the same dimension.

In order to measure the size of the errors, we define

$$\|\mathbf{a}\|_\infty = \max_{1\le j\le p} |a_j|, \quad \mathbf{a} = (a_1, a_2, \ldots, a_p) \in \mathbb{R}^p, \quad \|\mathbf{V}\|_1 = \sum_{j=0}^{J} h\,|V_j|, \quad \mathbf{V} \in \mathbb{R}^{J+1}.$$

Now, we endow the spaces X_h and Y_h with the following norms. If $\left(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N\right) \in X_h$, then

$$\| \left(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N\right) \|_{X_h} = \max_{0\le n\le N} \|\mathbf{V}^n\|_\infty.$$

On the other hand, if $\left(\mathbf{P}^0, \mathbf{P}_0, \mathbf{P}^1, \mathbf{P}^2, \ldots, \mathbf{P}^N\right) \in Y_h$, then

$$\| \left(\mathbf{P}^0, \mathbf{P}_0, \mathbf{P}^1, \mathbf{P}^2, \ldots, \mathbf{P}^N\right) \|_{Y_h} = \|\mathbf{P}^0\|_\infty + \|\mathbf{P}_0\|_\infty + \sum_{n=1}^{N} h\,\|\mathbf{P}^n\|_\infty$$

Let u represent the solution of (2.1)-(2.4). For each $h \in H$ we define

$$\mathbf{u}_h = (\mathbf{u}^0, \mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^N), \quad \mathbf{u}^n = (u_0^n, u_1^n, \ldots, u_J^n) \in \mathbb{R}^{J+1},$$

$$u_j^n = u(a_j, t_n), \quad 0 \le j \le J, \quad 0 \le n \le N.$$

Let be M a fixed positive constant and denote by $B_{X_h}(\mathbf{u}_h, Mh) \subset X_h$ the open ball with center $\mathbf{u}_h$ and radius Mh. Next, we introduce the mapping

$$\mathbf{\Phi}_h : B_{X_h}(\mathbf{u}_h, Mh) \to Y_h,$$

$$\mathbf{\Phi}_h \left(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N\right) = \left(\mathbf{P}^0, \mathbf{P}_0, \mathbf{P}^1, \ldots, \mathbf{P}^N\right), \tag{4.1}$$

defined by the following equations:

$$\mathbf{P}^0 = \mathbf{V}^0 - \mathbf{U}^0 \in \mathbb{R}^{J+1}, \tag{4.2}$$

$$P_0^n = V_0^n - Q_h\left(\boldsymbol{\alpha}(\mathbf{V}^n)\,\mathbf{V}^n\right), \quad 1 \le n \le N, \tag{4.3}$$

and for $0 \le n \le N-1$,

$$P_{j+1}^{n+1} = \frac{V_{j+1}^{n+1} - V_j^n \, \exp\left(-h \, \left[m\left(a_{j+\frac{1}{2}}\right) + \mu\left(a_{j+\frac{1}{2}}, Q_h^*(\boldsymbol{\gamma}_\mu \, \mathbf{V}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}}\right)\right]\right)}{h}, \; 0 \le j \le J^* - 1, \quad (4.4)$$

$$P_{j+1}^{n+1} = \frac{V_{j+1}^{n+1} - V_j^n \, e^{[f(a_j) - f(a_{j+1})]} \, \exp\left(-h \, \mu\left(a_{j+\frac{1}{2}}, Q_h^*(\boldsymbol{\gamma}_\mu \, \mathbf{V}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}}\right)\right)}{h}, \quad J^* \le j \le J - 1, \quad (4.5)$$

where

$$V_{j+1}^{n+\frac{1}{2}} = V_j^n \, \exp\left(-\frac{h}{2} \, \left[m\left(a_j\right) + \mu\left(a_j, Q_h(\boldsymbol{\gamma}_\mu \, \mathbf{V}^n), t_n\right)\right]\right), \qquad 0 \le j \le J^* - 1, \tag{4.6}$$

$$V_{j+1}^{n+\frac{1}{2}} = V_j^n \, e^{[f(a_j) - f(a_{j+1/2})]} \, \exp\left(-\frac{h}{2} \, \mu\left(a_j, Q_h(\boldsymbol{\gamma}_\mu \, \mathbf{V}^n), t_n\right)\right), \quad J^* \le j \le J - 1, \tag{4.7}$$

and the vector $\mathbf{V}^0$ represents an approximation to the theoretical solution at $t = 0$, and we consider the same notation as in Section 3.

It is clear that $\left(\mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^N\right) \in X_h$ is a solution of (3.3)-(3.5) if and only if

$$\boldsymbol{\Phi}_h\left(\mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^N\right) = \mathbf{0}. \tag{4.8}$$

We begin with the following auxiliary result.

Proposition 1 *Assume hypotheses (H1)-(H5). Let be* $\mathbf{V}^n, \mathbf{W}^n \in B_\infty(\mathbf{u}^n, Mh), \quad 1 \le n \le N$. *Then, as* $h \to 0$, *the following hold:*

$$|Q_h(\boldsymbol{\gamma}_\phi \, \mathbf{V}^n) - Q_h(\boldsymbol{\gamma}_\phi \, \mathbf{W}^n)| \le C \, \|\mathbf{V}^n - \mathbf{W}^n\|_1, \quad 1 \le n \le N, \tag{4.9}$$

$$|Q_h(\boldsymbol{\alpha}(\mathbf{V}^n) \, \mathbf{V}^n) - Q_h(\boldsymbol{\alpha}(\mathbf{W}^n) \, \mathbf{W}^n)| \le C \, \|\mathbf{V}^n - \mathbf{W}^n\|_1, \quad 1 \le n \le N, \tag{4.10}$$

$$|V_{j+1}^{n+\frac{1}{2}} - W_{j+1}^{n+\frac{1}{2}}| \le |V_j^n - W_j^n| + C \, h \, \|\mathbf{V}^n - \mathbf{W}^n\|_1, \quad 0 \le j \le J - 1, \, 1 \le n \le N - 1, \tag{4.11}$$

$$|Q_h^*(\boldsymbol{\gamma}_\phi \, \mathbf{V}^{n+\frac{1}{2}}) - Q_h^*(\boldsymbol{\gamma}_\phi \, \mathbf{W}^{n+\frac{1}{2}})| \le C \, \|\mathbf{V}^n - \mathbf{W}^n\|_1, \quad 1 \le n \le N - 1, \tag{4.12}$$

for $\phi = \mu$, α.

PROOF. It follows from (3.8) and hypothesis (H2) that

$$|Q_h(\boldsymbol{\gamma}_\phi \, \mathbf{V}^n) - Q_h(\boldsymbol{\gamma}_\phi \, \mathbf{W}^n)| = |Q_h(\boldsymbol{\gamma}_\phi \, (\mathbf{V}^n - \mathbf{W}^n))| \le C \, \sum_{i=0}^{J} h \, |V_i^n - W_i^n| = C \, \|\mathbf{V}^n - \mathbf{W}^n\|_1,$$

$1 \le n \le N$. We omit the proof of (4.10) because it is analogous to the one just presented. Next, from (4.6), we obtain for $0 \le j \le J^* - 1$,

$$V_{j+1}^{n+\frac{1}{2}} - W_{j+1}^{n+\frac{1}{2}} = (V_j^n - W_j^n)\,\exp\left(-\frac{h}{2}\,\left[m\,(a_j) + \mu\left(a_j, Q_h(\boldsymbol{\gamma}_\mu\,\mathbf{V}^n), t_n\right)\right]\right) \tag{4.13}$$
$$+ W_j^n\,\exp\left(-\frac{h}{2}\,m\,(a_j)\right)\left[\exp\left(-\frac{h}{2}\,\mu\left(a_j, Q_h(\boldsymbol{\gamma}_\mu\,\mathbf{V}^n), t_n\right)\right) - \exp\left(-\frac{h}{2}\,\mu\left(a_j, Q_h(\boldsymbol{\gamma}_\mu\,\mathbf{W}^n), t_n\right)\right)\right].$$

Also, using (4.7), we have for $J^* \leq j \leq J-1$,

$$V_{j+1}^{n+\frac{1}{2}} - W_{j+1}^{n+\frac{1}{2}} = (V_j^n - W_j^n)\,e^{[f(a_j)-f(a_{j+1/2})]}\,\exp\left(-\frac{h}{2}\mu\left(a_j, Q_h(\boldsymbol{\gamma}_\mu\,\mathbf{V}^n), t_n\right)\right)$$
$$+ W_j^n\,e^{[f(a_j)-f(a_{j+1/2})]}\left[\exp\left(-\frac{h}{2}\,\mu\left(a_j, Q_h(\boldsymbol{\gamma}_\mu\,\mathbf{V}^n), t_n\right)\right) - \exp\left(-\frac{h}{2}\,\mu\left(a_j, Q_h(\boldsymbol{\gamma}_\mu\,\mathbf{W}^n), t_n\right)\right)\right].$$

Now, the regularity hypotheses (H1)-(H5), inequality (4.9) and the relation $\|\mathbf{W}^n\|_\infty \leq C$ yield, for $0 \leq j \leq J-1,\ 0 \leq n \leq N-1$,

$$\begin{aligned}\left|V_{j+1}^{n+\frac{1}{2}} - W_{j+1}^{n+\frac{1}{2}}\right| &\leq |V_j^n - W_j^n| + C\,h\,\left|Q_h(\boldsymbol{\gamma}_\mu\,\mathbf{V}^n) - Q_h(\boldsymbol{\gamma}_\mu\,\mathbf{W}^n)\right| \\ &\leq |V_j^n - W_j^n| + C\,h\,\|\mathbf{V}^n - \mathbf{W}^n\|_1.\end{aligned} \tag{4.14}$$

Finally, using (3.9), the inequality (4.11) and the hypothesis H2, we see that, for $1 \leq n \leq N-1$,

$$\begin{aligned}&\left|Q_h^*\left(\boldsymbol{\gamma}_\phi\,\mathbf{V}^{n+\frac{1}{2}}\right) - Q_h^*\left(\boldsymbol{\gamma}_\phi\,\mathbf{W}^{n+\frac{1}{2}}\right)\right| = \left|Q_h^*\left(\boldsymbol{\gamma}_\phi\left[\mathbf{V}^{n+\frac{1}{2}} - \mathbf{W}^{n+\frac{1}{2}}\right]\right)\right| \\ &\leq C\sum_{i=1}^{J} h\left|V_i^{n+\frac{1}{2}} - W_i^{n+\frac{1}{2}}\right| \leq C\sum_{i=0}^{J-1} h\left(|V_i^n - W_i^n| + Ch\,\|\mathbf{V}^n - \mathbf{W}^n\|_1\right) \\ &\leq C\,\|\mathbf{V}^n - \mathbf{W}^n\|_1,\end{aligned}$$

as desired. □

The next result shows that operator defined by (4.1) is well defined.

Proposition 2 *Assume that hypotheses (H1)-(H5) hold. If*

$$\left(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N\right) \in B_{X_h}(\mathbf{u}_h, Mh),$$

where M is a fixed positive constant, then, for h sufficiently small,

$$Q_h(\boldsymbol{\gamma}_\phi\,\mathbf{V}^n) \in D_\phi, \quad 0 \leq n \leq N, \quad \phi = \mu, \alpha, \tag{4.15}$$

and

$$Q_h^*\left(\boldsymbol{\gamma}_\phi\,\mathbf{V}^{n+\frac{1}{2}}\right) \in D_\phi, \quad 0 \leq n \leq N-1, \quad \phi = \mu, \alpha. \tag{4.16}$$

PROOF. Definition (3.8), hypotheses (H1)-(H5), the inequality (4.9), and the fact that $\mathbf{V}^n$ is bounded, allow us to obtain, for $0 \le n \le N$ and $h \to 0$,

$$\begin{aligned} \left|Q_h(\boldsymbol{\gamma}_\phi \mathbf{V}^n) - I_\phi(t_n)\right| &\le \left|Q_h(\boldsymbol{\gamma}_\phi \mathbf{V}^n) - Q_h(\boldsymbol{\gamma}_\phi \mathbf{u}^n)\right| + \left|Q_h(\boldsymbol{\gamma}_\phi \mathbf{u}^n) - I_\phi(t_n)\right| \\ &\le \|\mathbf{V}^n - \mathbf{u}^n\|_1 + o(1). \end{aligned} \tag{4.17}$$

Therefore, (4.15) holds for h sufficiently small. On the other hand, we can see from (3.9), hypotheses (H1)-(H5), inequality (4.12), property

$$\left|u\left(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}\right) - u_{j+1}^{n+\frac{1}{2}}\right| = o(1), \quad 0 \le j \le J-1, \tag{4.18}$$

and the boundedness of $\mathbf{V}^n$, that for $0 \le n \le N-1$,

$$\begin{aligned} \left|Q_h^*\left(\boldsymbol{\gamma}_\phi \mathbf{V}^{n+\frac{1}{2}}\right) - I_\phi\left(t_{n+\frac{1}{2}}\right)\right| &\le \left|Q_h^*\left(\boldsymbol{\gamma}_\phi \mathbf{V}^{n+\frac{1}{2}}\right) - Q_h^*\left(\boldsymbol{\gamma}_\phi \mathbf{u}^{n+\frac{1}{2}}\right)\right| + \left|Q_h^*\left(\boldsymbol{\gamma}_\phi \mathbf{u}^{n+\frac{1}{2}}\right) - I_\phi\left(t_{n+\frac{1}{2}}\right)\right| \\ &\le \|\mathbf{V}^n - \mathbf{u}^n\|_1 + o(1) \end{aligned} \tag{4.19}$$

holds as $h \to 0$, where $\mathbf{u}^{n+\frac{1}{2}}$ is defined in (4.6)-(4.7)). Therefore, (4.16) holds for h sufficiently small. □

Now we define *the local discretization error* as

$$\mathbf{l}_h = \boldsymbol{\Phi}_h(\mathbf{u}_h) \in Y_h,$$

and we say that the discretization (4.1) is *consistent* if

$$\lim_{h\to 0} \|\boldsymbol{\Phi}_h(\mathbf{u}_h)\|_{Y_h} = \lim_{h\to 0} \|\mathbf{l}_h\|_{Y_h} = 0.$$

The next theorem establishes the consistency of the numerical method defined by (3.3)-(3.5).

Theorem 3 *Assume that hypotheses (H1)-(H5) hold. Then, as $h \to 0$, the local discretization error satisfies,*

$$\|\boldsymbol{\Phi}_h(\mathbf{u}_h)\|_{Y_h} = \|\mathbf{u}^0 - \mathbf{U}^0\|_\infty + O(h^2). \tag{4.20}$$

PROOF. Let us denote $\boldsymbol{\Phi}_h(\mathbf{u}_h) = (\mathbf{L}^0, \mathbf{L}_0, \mathbf{L}^1, \mathbf{L}^2, \ldots, \mathbf{L}^N)$. First we set the bounds for $\mathbf{L}^{n+1}$, $0 \le n \le N-1$. Using (3.2) and (4.4), the regularity hypotheses (H1)-(H5), and the standard error bound of the mid-point quadrature rule, we have for $0 \le j \le J^* - 1$,

$$
\begin{aligned}
\left|L_{j+1}^{n+1}\right| \leq & \frac{|u_j^n|}{h}\left\{\left|\exp\left(-\int_0^h \left[m\left(a_j+\sigma\right)+\mu\left(a_j+\sigma, I_\mu(t_n+\sigma), t_n+\sigma\right)\right] d\sigma\right)\right.\right. \\
& \left. - \exp\left(-h\left[m\left(a_{j+\frac{1}{2}}\right)+\mu\left(a_{j+\frac{1}{2}}, I_\mu\left(t_{n+\frac{1}{2}}\right), t_{n+\frac{1}{2}}\right)\right]\right)\right| \\
& + \left|\exp\left(-h\left[m\left(a_{j+\frac{1}{2}}\right)+\mu\left(a_{j+\frac{1}{2}}, I_\mu\left(t_{n+\frac{1}{2}}\right), t_{n+\frac{1}{2}}\right)\right]\right)\right. \\
& \left.\left. - \exp\left(-h\left[m\left(a_{j+\frac{1}{2}}\right)+\mu\left(a_{j+\frac{1}{2}}, Q_h^*(\boldsymbol{\gamma}_\mu\, \mathbf{u}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}}\right)\right]\right)\right|\right\} \\
\leq & C\left\{h^2+\left|\mu\left(a_{j+\frac{1}{2}}, I_\mu\left(t_{n+\frac{1}{2}}\right), t_{n+\frac{1}{2}}\right)-\mu\left(a_{j+\frac{1}{2}}, Q_h^*(\boldsymbol{\gamma}_\mu\, \mathbf{u}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}}\right)\right|\right\} \\
\leq & C\left\{h^2+\left|I_\mu\left(t_{n+\frac{1}{2}}\right)-Q_h^*(\boldsymbol{\gamma}_\mu\, \mathbf{u}^{n+\frac{1}{2}})\right|\right\}, \qquad (4.21)
\end{aligned}
$$

where $\mathbf{u}^{n+\frac{1}{2}}$ are computed by means of (4.6)-(4.7) and, therefore, they are not the values of the solution at $t_{n+\frac{1}{2}}$. Following an analogous argument we can derive, for $J^* \leq j \leq J-1$,

$$
\left|L_{j+1}^{n+1}\right| \leq C\left\{h^2+\left|I_\mu\left(t_{n+\frac{1}{2}}\right)-Q_h^*(\boldsymbol{\gamma}_\mu\, \mathbf{u}^{n+\frac{1}{2}}))\right|\right\}. \qquad (4.22)
$$

Also, from the convergence properties of the quadrature rules employed, we obtain the following bounds:

$$
\left|u\left(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}\right)-u_j^n \exp\left(-\frac{h}{2}\left[m\left(a_j\right)+\mu\left(a_j, Q_h(\boldsymbol{\gamma}_\mu\, \mathbf{u}^n), t_n\right)\right]\right)\right| \leq C\, h^2,\ 0 \leq j \leq J^*-1, \quad (4.23)
$$

$$
\left|u\left(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}\right)-u_j^n\, e^{[f(a_j)-f(a_{j+1/2})]} \exp\left(-\frac{h}{2}\, \mu\left(a_j, Q_h(\boldsymbol{\gamma}_\mu\, \mathbf{u}^n), t_n\right)\right)\right| \leq C\, h^2,\ J^* \leq j \leq J-1, \quad (4.24)
$$

that lead to the relation

$$
\left|I_\mu\left(t_{n+\frac{1}{2}}\right)-Q_h^*(\boldsymbol{\gamma}_\mu\, \mathbf{u}^{n+\frac{1}{2}})\right| \leq C\, h^2, \qquad (4.25)
$$

which can be substituted in (4.21) and (4.22) to see that

$$
|L_{j+1}^{n+1}| \leq C\, h^2, \quad 0 \leq j \leq J-1. \qquad (4.26)
$$

Now we consider bounding $\mathbf{L}_0$. Using of (4.3), the regularity hypotheses (H1)-(H5), the standard error bounds for the quadrature rule, and similar arguments to those employed above, we have for $1 \leq n \leq N$,

$$
|L_0^n| \leq \left|\int_0^{a_\dagger} \alpha(a, I_\alpha(t_n), t_n)\, u(a, t_n)\, da - Q_h\left(\boldsymbol{\alpha}(\mathbf{u}^n)\, \mathbf{u}^n\right)\right| \leq C\, h^2+\left|I_\alpha\left(t_n\right)-Q_h(\boldsymbol{\gamma}_\alpha\, \mathbf{u}^n)\right| \leq C\, h^2. \qquad (4.27)
$$

This completes the proof of (4.20). □

Another notion that plays an important role in the analysis of the numerical method is that of *stability with h-dependent thresholds*. For $h \in H$, let $M_h > 0$ be a real number (*the stability threshold*); we say that the discretization (4.1) is *stable* for $\mathbf{u}$ restricted to the thresholds M_h, if there exist two positive constants h_0 and S (*the stability constant*) such that, for any $h \in H$ with $h \leq h_0$, the open ball $B_{X_h}(\mathbf{u}_h, M_h)$ is contained in the domain of $\mathbf{\Phi}_h$ and, for all $\mathbf{V}$, $\mathbf{W}_h$ in that ball,

$$\|\mathbf{V} - \mathbf{W}_h\| \leq S\, \|\mathbf{\Phi}_h(\mathbf{V}_h) - \mathbf{\Phi}_h(\mathbf{W}_h)\|.$$

Next, we introduce a theorem that establishes the *stability* of the discretization defined by equations (3.3)-(3.5).

Theorem 4 *Assume that hypotheses (H1)-(H5) hold and let $M > 0$ be a fixed constant. Then, the discretization (3.3)-(3.5) is estable for $\mathbf{u}_h$ with thresholds Mh.*

PROOF. Let $\left(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N\right)$, $\left(\mathbf{W}^0, \mathbf{W}^1, \ldots, \mathbf{W}^N\right) \in B_{X_h}(\mathbf{u}_h, Mh)$ and set

$$\mathbf{E}^n = \mathbf{V}^n - \mathbf{W}^n \in \mathbb{R}^{J+1}, \quad 0 \leq n \leq N,$$

$$\mathbf{\Phi}_h\left(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N\right) = \left(\mathbf{P}^0, \mathbf{P}_0, \mathbf{P}^1, \mathbf{P}^2, \ldots, \mathbf{P}^N\right),$$
$$\mathbf{\Phi}_h\left(\mathbf{W}^0, \mathbf{W}^1, \ldots, \mathbf{W}^N\right) = \left(\mathbf{R}^0, \mathbf{R}_0, \mathbf{R}^1, \mathbf{R}^2, \ldots, \mathbf{R}^N\right).$$

From (4.4) we have, for $0 \leq j \leq J^* - 1$,

$$\begin{aligned} E_{j+1}^{n+1} = & E_j^n\, \exp\left(-h\,\left[m\left(a_{j+\frac{1}{2}}\right) + \mu\left(a_{j+\frac{1}{2}}, Q_h^*(\boldsymbol{\gamma}_\mu\, \mathbf{V}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}}\right)\right]\right) \\ & + W_j^n\, \exp\left(-h\, m\left(a_{j+\frac{1}{2}}\right)\right)\left(\exp\left(-h\,\mu\left(a_{j+\frac{1}{2}}, Q_h^*(\boldsymbol{\gamma}_\mu\, \mathbf{V}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}}\right)\right)\right. \\ & \left. - \exp\left(-h\,\mu\left(a_{j+\frac{1}{2}}, Q_h^*(\boldsymbol{\gamma}_\mu\, \mathbf{W}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}}\right)\right)\right) + h\,(P_{j+1}^{n+1} - R_{j+1}^{n+1}), \end{aligned} \tag{4.28}$$

and from (4.5) we have, for $J^* \leq j \leq J - 1$,

$$\begin{aligned} E_{j+1}^{n+1} = & E_j^n\, e^{[f(a_j) - f(a_{j+1})]}\, \exp\left(-h\,\mu\left(a_{j+\frac{1}{2}}, Q_h^*(\boldsymbol{\gamma}_\mu\, \mathbf{V}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}}\right)\right) \\ & + W_j^n\, e^{[f(a_j) - f(a_{j+1})]}\, \left[\exp\left(-h\,\mu\left(a_{j+\frac{1}{2}}, Q_h^*(\boldsymbol{\gamma}_\mu\, \mathbf{V}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}}\right)\right)\right. \\ & \left. - \exp\left(-h\,\mu\left(a_{j+\frac{1}{2}}, Q_h^*(\boldsymbol{\gamma}_\mu\, \mathbf{W}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}}\right)\right)\right] + h\left(P_{j+1}^{n+1} - R_{j+1}^{n+1}\right). \end{aligned} \tag{4.29}$$

The regularity hypotheses (H1)-(H5), formulae (4.28)-(4.29), inequality (4.12), and $\|\mathbf{W}\|_\infty \leq C$ imply that, for $0 \leq j \leq J - 1$,

$$\begin{aligned} |E_{j+1}^{n+1}| & \leq |E_j^n| + C\,h\,\left|Q_h^*\left(\boldsymbol{\gamma}_\mu\, \mathbf{V}^{n+\frac{1}{2}}\right) - Q_h^*\left(\boldsymbol{\gamma}_\mu\, \mathbf{W}^{n+\frac{1}{2}}\right)\right| + h\left|P_{j+1}^{n+1} - R_{j+1}^{n+1}\right| \\ & \leq |E_j^n| + C\,h\,\|\mathbf{E}^n\|_1 + h\left|P_{j+1}^{n+1} - R_{j+1}^{n+1}\right|. \end{aligned} \tag{4.30}$$

Thus, when $N \geq n > j \geq 1$, from (4.30) we have

$$
\begin{aligned}
|E_j^n| &\le |E_0^{n-j}| + C\,h \sum_{l=1}^{j} \left\|\mathbf{E}^{n-l}\right\|_1 + h \sum_{l=0}^{j-1} |P_{j-l}^{n-l} - R_{j-l}^{n-l}| \\
&\le |E_0^{n-j}| + C\,h \sum_{l=0}^{n-1} \left\|\mathbf{E}^{l}\right\|_1 + h \sum_{l=1}^{n} \|\mathbf{P}^l - \mathbf{R}^l\|_\infty .
\end{aligned} \tag{4.31}
$$

On the other hand, when $j > n \ge 1$, from (4.30) we obtain

$$
\begin{aligned}
|E_j^n| &\le |E_{j-n}^{0}| + C\,h \sum_{l=1}^{n} \left\|\mathbf{E}^{n-l}\right\|_1 + h \sum_{l=0}^{n-1} |P_{j-l}^{n-l} - R_{j-l}^{n-l}| \\
&\le |E_{n-j}^{0}| + C\,h \sum_{l=0}^{n-1} \left\|\mathbf{E}^{l}\right\|_1 + h \sum_{l=1}^{n} \|\mathbf{P}^l - \mathbf{R}^l\|_\infty .
\end{aligned} \tag{4.32}
$$

Now, by (4.3)

$$
\begin{aligned}
E_0^n &= Q_h\Big(\boldsymbol{\alpha}(\mathbf{V}^n)\,\mathbf{V}^n\Big) - Q_h\Big(\boldsymbol{\alpha}(\mathbf{W}^n)\,\mathbf{W}^n\Big) + (P_0^n - R_0^n) \\
&= Q_h\Big(\boldsymbol{\alpha}(\mathbf{V}^n)\,\mathbf{E}^n\Big) + Q_h\Big([\boldsymbol{\alpha}(\mathbf{V}^n) - \boldsymbol{\alpha}(\mathbf{W}^n)]\;\mathbf{W}^n\Big) + (P_0^n - R_0^n).
\end{aligned} \tag{4.33}
$$

Also, using hypotheses (H1)-(H5), inequalities (4.9) and (4.10), and $\|\mathbf{W}^n\|_\infty \le C$, we have

$$
\begin{aligned}
|E_0^n| &\le \|\mathbf{E}^n\|_1 + C\;|Q_h(\boldsymbol{\gamma}_\alpha\,\mathbf{V}^n) - Q_h(\boldsymbol{\gamma}_\alpha\,\mathbf{W}^n)| + |P_0^n - R_0^n| \\
&\le C\,\|\mathbf{E}^n\|_1 + |P_0^n - R_0^n|.
\end{aligned} \tag{4.34}
$$

Next, multiplying $|E_j^n|$ by h and summing in j, $0 \le j \le J$, from (4.31), (4.32), and (4.34) we have, for $1 \le n \le N$,

$$
\begin{aligned}
\|\mathbf{E}^n\|_1 &= h\,|E_0^n| + \sum_{j=1}^{n-1} h\,|E_j^n| + \sum_{j=n}^{J} h\,|E_j^n| \\
&\le h\,\left(C\,\|\mathbf{E}^n\|_1 + |P_0^n - R_0^n|\right) + \sum_{j=1}^{n-1} h\,\left(|E_0^{n-j}| + C\,h \sum_{l=0}^{n-1} \left\|\mathbf{E}^l\right\|_1 + h \sum_{l=1}^{n} \|\mathbf{P}^l - \mathbf{R}^l\|_\infty\right) \\
&\quad + \sum_{j=n}^{J} h\,\left(|E_{n-j}^0| + C\,h \sum_{l=0}^{n-1} \left\|\mathbf{E}^l\right\|_1 + h \sum_{l=1}^{n} \|\mathbf{P}^l - \mathbf{R}^l\|_\infty\right) \\
&\le C\,\|\mathbf{E}^0\|_1 + h\,\left(C\,\|\mathbf{E}^n\|_1 + |P_0^n - R_0^n|\right) + \sum_{j=1}^{n-1} h\,\left(C\,\|\mathbf{E}^{n-j}\|_1 + |P_0^{n-j} - R_0^{n-j}|\right) \\
&\quad + C\,h \sum_{l=0}^{n-1} \left\|\mathbf{E}^l\right\|_1 + C\,h \sum_{l=1}^{n} \|\mathbf{P}^l - \mathbf{R}^l\|_\infty \\
&\le C\,\|\mathbf{E}^0\|_1 + C\,h \sum_{l=0}^{n} \left\|\mathbf{E}^l\right\|_1 + C \sum_{l=1}^{n} h\,|P_0^l - R_0^l| + C\,h \sum_{l=1}^{n} \|\mathbf{P}^l - \mathbf{R}^l\|_\infty .
\end{aligned}
$$

Using the discrete Gronwall lemma, it follows that

$$\|\mathbf{E}^n\|_1 \leq C\left(\|\mathbf{E}^0\|_1 + \|\mathbf{P}_0 - \mathbf{R}_0\|_\infty + h\sum_{l=1}^{n}\|\mathbf{P}^l - \mathbf{R}^l\|_\infty\right). \tag{4.35}$$

Then, we substitute (4.35) in (4.31) and (4.34) to conclude the proof. □

Now, we define the *global discretization error* as

$$\mathbf{e}_h = \mathbf{u}_h - \mathbf{U}_h \in X_h.$$

We say that the discretization (4.1) is *convergent* if there exists $h_0 > 0$ such that, for each $h \in H$ with $h \leq h_0$, (4.8) has a solution $\mathbf{U}_h$ for which

$$\lim_{h\to 0}\|\mathbf{u}_h - \mathbf{U}_h\|_{X_h} = \lim_{h\to 0}\|\mathbf{e}_h\|_{X_h} = 0.$$

To conclude our convergence analysis we shall use the following result from the general discretization framework introduced by López-Marcos *et al.* [12].

Theorem 5 *Assume that (4.1) is consistent and stable with thresholds M_h. If Φ_h is continuous in $B(\mathbf{u}_h, M_h)$ and $\|\mathbf{l}_h\|_{Y_h} = o(M_h)$ as $h \to 0$, then for h sufficiently small,*

i) the discrete equations (4.8) possess a unique solution in $B(\mathbf{u}_h, M_h)$;

ii) the solutions converge and $\|\mathbf{e}_h\|_{X_h} = O(\|\mathbf{l}_h\|_{Y_h})$.

Finally, we state the theorem that establishes the *convergence* of our numerical method defined by equations (3.3)-(3.5).

Theorem 6 *Assume that hypotheses (H1)-(H5) hold. Then, for h sufficiently small, the numerical method defined by (3.3)-(3.5) has a unique solution $\mathbf{U}_h \in B_{X_h}(\mathbf{u}_h, Mh)$ and*

$$\|\mathbf{U}_h - \mathbf{u}_h\|_{X_h} \leq C\left[\|\mathbf{u}^0 - \mathbf{U}^0\|_\infty + O(h^2)\right].$$

The proof of Theorem 6 is immediate from the consistency—Theorem 3, the stability —Theorem 4, and Theorem 5.

5 Numerical results and conclusions

We have carried out several numerical experiments using the algorithm defined in Section 3. We considered different test problems presenting meaningful nonlinearities that appear in the literature,

[11]. The numerical integration for each numerical experiment was carried out over the time interval $[0,1]$. In all the simulations we used the parameter values $a_{\dagger}=1$, $A^{*}=0.9$.

Problem 1. This is one of the examples present in [11], using the fertility and mortality rates $\alpha(a,z,t)=4$, $m(a)=\dfrac{1}{1-a}$ and $\mu(a,z,t)=z$. The weight functions are taken as $\gamma_{\mu}\equiv\gamma_{\alpha}\equiv 1$, and we consider as initial age density the function $u_0(a)=4\,(1-a)\,e^{-\lambda\,a}$, where $\lambda=2.5569290855$. Problem (2.1)-(2.4) then has the following solution,

$$u(a,t)=4\,(1-a)\,e^{-\lambda\,a}\,\frac{\lambda}{(\lambda-1)\,e^{-\lambda\,t}+1}.$$

Problem 2. We take now $m(a)=\dfrac{0.5}{1-a}$ and the other functions as in **Problem 1**. Problem (2.1)-(2.4) now has the following solution,

$$u(a,t)=4\,(1-a)\,e^{-\lambda\,a}\,\frac{\lambda}{(\lambda-1)\,e^{-\lambda\,t}+1},$$

where $\lambda=3.22540174092$.

Problem 3. In this case, we chose $\mu(a,z,t)=z^2$, the other functions as in **Problem 2**. The solution to problem (2.1)-(2.4) is then given by

$$u(a,t)=4\,(1-a)\,e^{-\lambda\,a}\,\sqrt{\frac{\lambda}{(\lambda-1)\,e^{-2\,\lambda\,t}+1}}.$$

Since we know the exact solution for each of these problems, we can show numerically that our method is second order accurate by means of error tables. In each table below the second column shows the global error for the total population computed as follows,

$$e_h=\max_{0\le n\le N}|Q_h(\mathbf{U}^n)-P(t_n)|.$$

The fourth column represents the global error for the age density of the population computed using the formula

$$e_h=\max_{0\le n\le N,0\le j\le J}|u(a_j,t_n)-U_j^n|.$$

The third and fifth columns display the experimental order of convergence of the method, s, computed as

$$s=\frac{\log(e_{2\,h}/e_h)}{\log(2)},$$

using the values of the second and forth column respectively. Each row of the tables corresponds to different value of the discretization parameter.

Table 1
Errors and convergence order for Problem 1.

k	$\max_{0\le n\le N} \lvert Q_h(\mathbf{U}^n) - P(t_n)\rvert$	Order	$\max_{0\le n\le N, 0\le j\le J} \lvert u(a_j, t_n) - U_j^n\rvert$	Order
1.25000E-02	6.164114E-03		2.465645E-02	
6.25000E-03	1.561714E-03	1.98	6.246857E-03	1.98
3.12500E-03	3.928375E-04	1.99	1.571350E-03	1.99
1.56250E-03	9.849879E-05	2.00	3.939952E-04	2.00
7.81250E-04	2.466012E-05	2.00	9.864050E-05	2.00
3.90625E-04	6.169412E-06	2.00	2.467765E-05	2.00

Table 2
Errors and convergence order for Problem 2.

k	$\max_{0\le n\le N} \lvert Q_h(\mathbf{U}^n) - P(t_n)\rvert$	Order	$\max_{0\le n\le N, 0\le j\le J} \lvert u(a_j, t_n) - U_j^n\rvert$	Order
1.25000E-02	7.745942E-03		3.098377E-02	
6.25000E-03	1.998691E-03	1.95	7.994764E-03	1.95
3.12500E-03	5.156830E-04	1.95	2.062732E-03	1.95
1.56250E-03	1.334500E-04	1.95	5.337998E-04	1.95
7.81250E-04	3.438362E-05	1.96	1.375345E-04	1.96
3.90625E-04	8.441683E-06	2.03	3.376673E-05	2.03

Table 3
Errors and convergence order for Problem 3.

k	$\max_{0\le n\le N} \lvert Q_h(\mathbf{U}^n) - P(t_n)\rvert$	Order	$\max_{0\le n\le N, 0\le j\le J} \lvert u(a_j, t_n) - U_j^n\rvert$	Order
1.25000E-02	3.417214E-03		1.366886E-02	
6.25000E-03	8.738994E-04	1.97	3.495597E-03	1.97
3.12500E-03	2.243750E-04	1.96	8.974998E-04	1.96
1.56250E-03	5.794316E-05	1.95	2.317727E-04	1.95
7.81250E-04	1.497745E-05	1.95	5.990980E-05	1.95
3.90625E-04	3.764917E-06	1.99	1.505967E-05	1.99

We have described a new numerical method to approximate solutions of the initial-boundary value problem for a nonlinear, age structured population model with finite maximum age. The method is based on quadratures of the integrals resulting from the explicit integration of the differential equation in the model along characteristics.

We have proved the second-order convergence of the method provided the analytical solution is sufficiently regular. This order of convergence seems like a good compromise between efficiency for long time simulations and regularity constraints on the coefficient functions.

The implementation of the method is very straightforward since it is explicit and uses fractional time steps in order to avoid iterations on the nonlinearities. The results in the tables above clearly confirm the theoretical second order of convergence.

We point out that the results for Problems 2 and 3 are a novelty because, as indicated in [10,11], the numerical methods that have been previously proposed to approximate the solution of this model do not converge at their optimal order for mortality functions $m(a) = \dfrac{c}{1-a}$, $c < 1$, such as the one we chose for these problems.

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