Optimal control problem on insect pest populations

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Abstract

In this article we present a model of insect infestation of grape vines and consider the optimal control of the pest through the use of egg pesticides. We show existence and uniqueness and present the results of some numerical simulations.

Keywords: Optimal control ; age-structured equations ; population dynamics.

1. Introduction.

The European Grapevine moth *Lobesia botrana* has been the most serious wine pest in Europe, North Africa and in many Asian countries since the end of the XVIII century [3, 4, 7, 8, 9]. Many interventions have been developed to control this pest, e.g. insecticides, insect growth regulators, and mating disruption [6, 7]. A thorough description and modeling of this issue through age-structured deterministic population models was carried out in [5]. The well-posedness of the model was established and the three different control strategies just described were presented therein as optimal control mathematical problems.

Let u^e, u^l and u^f denote, respectively, the age density of the egg, larval and female populations. Their dynamics under egg pesticide control is modeled by :

$$\begin{cases} \frac{\partial u^e}{\partial t}(t,a) + \frac{\partial u^e}{\partial a}(t,a) = -\beta^e(a)u^e(t,a) - m^e(a)u^e(t,a), \quad a \in [0, L^e] \\ \frac{\partial u^l}{\partial t}(t,a) + \frac{\partial u^l}{\partial a}(t,a) = -\beta^l(a)u^l(t,a) - m^l(a)u^l(t,a), \quad a \in [0, L^l] \\ \frac{\partial u^f}{\partial t}(t,a) + \frac{\partial u^f}{\partial a}(t,a) = -m^f(a)u^f(t,a), \quad a \in [0, L^f] \end{cases} \tag{1}$$

$$\begin{cases} u^e(t,0) = \int_0^{L^f} \beta^f(a)u^f(t,a)da - v(t), \\ u^l(t,0) = \int_0^{L^e} \beta^e(a)u^e(t,a)da, \\ u^f(t,0) = \int_0^{L^l} \beta^l(a)u^l(t,a)da, \end{cases} \tag{2}$$

$$u^k(0,a) = u_0^k(a), \quad a \in [0, L^k], \quad k = e, l, f. \tag{3}$$

where v is the optimal control. The k stage mortality functions m^k, for k equals to e, l, f, are non negative, bounded functions which satisfy the condition

$$\lim_{a \to L^k} \int_0^a m^k(s)ds = \infty, \quad k = e, l, f, m.$$

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The function β^e models the transition function between the egg and the larval stage, whereas the function β^l models the transition function between the larval and the female stage. The function β^f corresponds to the birth function. These last three functions are non negative and bounded such that

$$\underline{\beta}^k \leq \beta^k(a) \leq \bar{\beta}^k, \quad k = e, l, f.$$

We are looking for the solution of the following optimal control problem

$$[\mathbf{P}] = \quad \min_{v \in K} \mathcal{J}(v) = \left[\eta \int_0^T v^2(t)dt + \mu \int_0^T \left(\int_0^{L^l} u^l(t,a)da \right)^2 dt \right],$$

where u^l is given by (1)-(2)-(3), η and μ are two positive constants and the set of admissible solutions is defined by

$$K = \{g(t) \in L^\infty([0,T]), 0 < \underline{g} \leq g \leq \bar{g}\}.$$

Barbu and Iannelli studied in [2] the problem of optimal control with the non linear structured population model, the Gurtin-MacCamy model. The control was applied on the vital rate and not on the egg renewal condition as on the problem [**P**].
In section 2 we compute upper and lower bounds of the control and study the well-posedness of [**P**]. The optimality conditions are explicitly given in section 3, and some numerical examples and conclusions aregiven in section 4.

2. Existence of the control v.

In this section, we start by computing the bounds of the control and then we prove, using minimizing sequences, the well posedness of [**P**].

To conserve the positivity of the biological system (1)-(2)-(3), the control v has to satisfy some conditions. The total number of newborns defined in (2) cannot be larger than the total number of newborns without control

$$0 \leq u^e(t,0) \leq \int_0^{L^f} \beta^f(a)u^f(t,a)da, \tag{4}$$

and is non negative for all time. Let $\underline{v}$ and $\bar{v}$ be the lower and the upper bounds of the control v, the first equation of (2) is bounded by

$$\underline{\beta}^f \int_0^{L^f} u^f(t,a)da - \bar{v} \leq u^e(t,0) \leq \bar{\beta}^f \int_0^{L^f} u^f(t,a)da - \underline{v}, \tag{5}$$

for all t. Let P be the total number of individuals in the population,

$$P(t) = P^e(t) + P^m(t) + P^f(t).$$

The integration of each differential equations of (1) on their age interval give the next equations

$$\partial_t P^k(t,a) + u^k(t,L^k) - u^k(t,0) = -\int_0^{L^k} \beta^k(a)u^k(t,a)da - \int_0^{L^k} m^k(a)u^k(t,a)da,$$

for the egg and larval stages ($k = e, l$) and

$$\partial_t P^f(t,a) + u^f(t, L^f) - u^f(t,0) = -\int_0^{L^f} m^f(a)u^f(t,a)da.$$

By summing these three equations and neglecting the negative terms we get

$$\partial_t P(t,a) \leq \int_0^{L^f} \beta^f(s)u^f(t,s)ds \leq \bar{\beta}^f P(t).$$

We apply the Gronwall Lemma on this last inequality to get an estimate on the total number of female

$$P^f(t) \leq P(t) \leq P(0)e^{\bar{\beta}^f T}, \tag{6}$$

useful to deduce, with (4) and (5) inequalities, the upper and lower bounds of the control

$$\bar{v} = \underline{\beta}^f P(0)e^{\bar{\beta}^f T}, \quad \underline{v} = 0.$$

Now, let d be the lower bound of the cost function $\mathcal{J}(v)$ and $\{v_n\}_n$, with n a non null integer, be a minimizing sequence of K such that

$$d < \mathcal{J}(v_n) \leq d + \frac{1}{n}.$$

The sequence $\{v_n\}_n$ is bounded in $L^2([0,T])$ space, as a consequence there exists a subsequence, named $\{v_{n_k}\}_k$, that converges weakly to the limit v^* of $L^2([0,T])$. The cost function defined in $\{v_n\}$ is dependent of the total number of larvae that is of the larval density function. Using the method of characteristics on the system (1)-(2)-(3) with $\{v_{n_k}\}_k$, we get explicit equations to compute the density functions of each populations,

$$u^j_{n_k}(t,a) = \begin{cases} u^j_0(a-t)e^{-\int_0^t \beta^j(s)+m^j(s)ds}, & a > t \\ u^j(t-a,0)e^{-\int_0^a \beta^j + m^j(s)(s)ds}, & a \leq t, \end{cases}$$

for $a \in [0, L^j]$, $t \in [0,T]$ and $j = e, l, f$. These sequences are bounded in $L^2([0,L^j] \times [0,T])$ space for j equals to e, l, f. We then can extract subsequences, respectively noted $\{\tilde{u}^e_{n_k}\}_k$, $\{\tilde{u}^l_{n_k}\}_k$ and $\{\tilde{u}^f_{n_k}\}_k$ that converge weakly to u^e_* , u^l_* and u^f_* in $L^2([0,L^j] \times [0,T])$ for j equals to e, l, f. The function of the total number of eggs, larvae and female defined by

$$P^j_{n_k}(t) = \int_0^{L^j} \tilde{u}^j_{n_k}(t,s)ds, \quad j = e, l, f,$$

is bounded. As computed in (6), the first derivative of the same functions satisfy the following inequalities

$$\partial_t P^j_{n_k}(t) \leq \int_0^{L^{\tilde{j}}} \beta^{\tilde{j}}(s)\tilde{u}^{\tilde{j}}_{n_k}(t,s)ds, \quad j = e, l, f, \quad \tilde{j} = f, e, l,$$

which prove that are all bounded. As a consequence, the functions $P^e_{n_k}$, $P^l_{n_k}$ and $P^f_{n_k}$ converge uniformly to P^e_*, P^l_*, P^f_*, that is, by the uniqueness of the limit, to

$$P^l_* = \int_0^{L^l} u^l_*(t,s)ds.$$

Finally, the cost function converges to

$$\mathcal{J}(v_{n_k}) = \eta \int_0^T (v_{n_k})^2(t)dt + \mu \int_0^T (\tilde{P}^l_{n_k}(t))^2 dt$$
$$\xrightarrow[k\to+\infty]{} \eta \int_0^T (v^*)^2(t)dt + \mu \int_0^T (P^l_*(t))^2 dt = \mathcal{J}(v^*) = d,$$

and the problem [**P**] admits at least one optimum.

3. Optimality conditions.

Here, we give the optimality conditions for the problem [**P**] via the study of the Lagrangian.

Let p^e, p^l and p^f be the dual variables. These functions satisfy the adjoint problem

$$\begin{cases} -\frac{\partial}{\partial t}p^e(t,a) - \frac{\partial}{\partial a}p^e(t,a) + \beta^e(a)p^e(t,a) + m^e(a)p^e - p^l(t,0)\beta^e(a) = 0, \\ -\partial_t p^l - \partial_a p^l + \beta^l p^l + m^l p^l - p^f(t,0)\beta^l(a) + 2\mu \int_0^{L^l} u^l da = 0, \\ -\frac{\partial}{\partial t}p^f(t,a) - \frac{\partial}{\partial a}p^f(t,a) + m^f(a)p^f(t,a) - p^e(t,0)\beta^f(a) = 0, \\ p^k(T,a) = 0, \quad k = e,l,f, \\ p^k(t,L^e) = 0, \quad k = e,l,f. \end{cases}$$

This system admits a unique solution [1] which is given below

$$p^e(t,a) = \begin{cases} \int_t^T \beta^e(a)p^l(s,0)e^{-\int_s^t(\beta^e+m^e)(\tau)d\tau}ds, & a \le t, \\ \int_a^{L^e} \beta^e(s)p^l(t,0)e^{-\int_s^a(\beta^e+m^e)(\tau)d\tau}ds, & a > t, \end{cases} \tag{7}$$

$$p^l(t,a) = \begin{cases} \int_t^T [\beta^l(a)p^f(s,0) + 2\mu \int_0^{L^l} u^l(s,x)dx]e^{-\int_s^t(\beta^l+m^l)(\tau)d\tau}ds, & a \le t, \\ \int_a^{L^l} [\beta^l(s)p^f(t,0) + 2\mu \int_0^{L^l} u^l(s+t-a,x)dx]e^{-\int_s^a(\beta^l+m^l)(\tau)d\tau}ds, & a > t, \end{cases} \tag{8}$$

$$p^f(t,a) = \begin{cases} \int_t^T \beta^f(a)p^e(s,0)e^{-\int_s^t m^f(\tau)d\tau}ds, & a \le t, \\ \int_a^{L^f} \beta^f(s)p^e(t,0)e^{-\int_s^a m^f(\tau)d\tau}ds, & a > t. \end{cases} \tag{9}$$

We remark that the dual variables are independent of the control v, the proof a unique solution to [**P**] is then more easier.

Theorem 1. *The problem [**P**] admits a unique optimum given by*

$$2\eta v^*(t) = -p^e(t,0).$$

where $p^e(t,0)$ is given by (7)-(8)-(9) and satisfies

$$-2\eta\underline{\beta}^f P(0)e^{\bar{\beta}^f T} \le p^e(t,0) \le 0.$$

4. Numerical examples.

We present two numerical applications of the problem $[P]$. In the first, the control is estimated for one insect generation whereas in the second example the control is computed for two insect generations.
The problem $[P]$ is solved with the Quasi-Newton algorithm as describe in [1, 5]. The partial differential equations (1)-(2)-(3) are approximated with a finite volume scheme.
The initial population is a cohort of 100 female of 1 day. The transition function between stages and the birth function are normal Gaussian of 7.5 expectancy and 0.35 standard deviation. The mortality functions are neglected here.
For one insect generation, the estimated control is drawn in green color on the upper graph (b) of the figure 1. The dark and blue curves are the total number of new eggs with respect to the time without and with egg pesticide control. The green curve of the lower graph (b) is the computed control for two insect generations. The fecundity, in this example, is the half of the first example. The quantity of egg pesticide predicted is more important in first generation than in the second.

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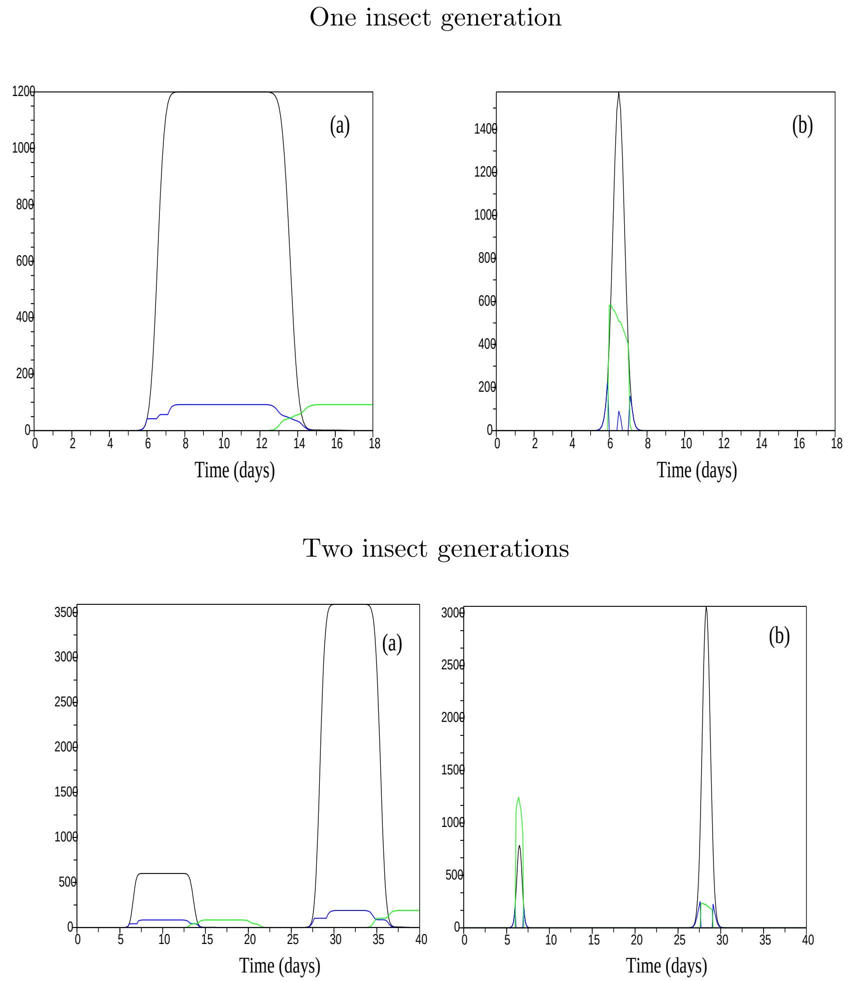

Fig. 1: (a) : Temporal eggs (black and blue curves) and larvae (green curve) dynamics. (b) : Eggs laying temporal dynamics and the estimated control v (green curve). The black and blue curves are the dynamic without and with egg pesticide control.