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The onset of oscillatory instability in a rotating layer of mercury heated from below and subject to a magnetic field

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A comparison is made between the predicted theoretical values of the Rayleigh number at the onset of instability in both the stationary and oscillatory modes with the observed experimental values of Nakagawa's (*Proc. R. Soc. Lond.* A **242**, 81–88 (1957)), when a layer of mercury is heated from below and simultaneously subjected to the effects of a magnetic field and rotation. The theoretical cell sizes and the gyration frequencies for the onset of oscillatory instabilities are also presented.

1. INTRODUCTION

The effects of a magnetic field and rotation acting simultaneously on a layer of mercury heated from below were investigated experimentally by Nakagawa (1957) and a qualitative comparison with linear convection theory was presented by Chandrasekhar (1961).

Chandrasekhar's (1961) linear treatment of the Rayleigh–Benard problem has shown the growth rate of infinitesimal disturbances to be always real, and hence that the principle of the exchange of stabilities is valid. A state is termed either stable or unstable depending on whether the growth rate of the disturbances is negative or positive. In the classical linear Rayleigh–Benard model, infinitesimal disturbances either decay or grow exponentially to finite amplitude, the governing equations of the system being time independent.

When rotation and/or a magnetic field are also present, the equations depend on the diffusivities of the fluid and, in these cases, the linear growth rate may have a non-zero complex component which permits an oscillatory bifurcation to take place. The occurrence of an oscillatory bifurcation is also termed overstability. Danielson (1961, 1963) attempted to use this mode of instability as a mechanism to describe sunspots, and predict their frequencies. In using a linear theory for nonlinear behaviour the assumptions that the lifetime of convection cells is proportional to the e-folding time of small velocity perturbations, and that the shape and dimensions of a convection cell remain the same as the velocity increases, need to be valid. More recently, Murphy & Steiner (1975) conducted a theoretical investigation which dealt with the onset of stationary convection quantitatively by applying boundary conditions which described the experimental situation of Nakagawa (1957) more closely than the artificial stress-free boundary conditions employed by Chandrasekhar (1961).

In this study, the magnetic and electric boundary conditions are chosen so as to represent more closely Nakagawa's (1957) results, and also to further extend Murphy & Steiner's (1975) work to deal with the onset of oscillatory convection.

Nakagawa's (1957) experiment investigated the effects that externally imposed rotation and an initially uniform vertical magnetic field had on the critical temperature gradient for the onset of thermal convective instability, both when this onset was steady or oscillatory, for various strengths of the imposed field while maintaining the rotation rate of the system as constant as possible. The experimental set-up corresponded to a thin 3 cm layer of mercury contained within a pyrex glass cylinder with a stainless steel bottom plate placed upon a heater and turntable, with the magnetic field supported by a 32.5 inch cyclotron magnet.

2. Mathematical parameterization

Rayleigh–Benard convection lends itself readily to a mathematical parameterization that works equally well for linear descriptions as it does for the finite amplitude situation. A description of the parameterization now follows.

The problem considered is the convective transport of heat through a horizontal layer of fluid which is heated from below, of infinite extent and depth d bound between two non-deformable, isothermal boundaries. The fluid has a coefficient of volume expansion α, viscosity μ, thermal conductivity K, specific heat at constant volume C_v, permeability μ^* and magnetic resistivity η. The layer, which is maintained at a temperature difference of ΔT, has an externally impressed uniform vertical magnetic field of strength H_0, and is given an angular speed of rotation, Ω_0, about a vertical axis.

The physics of such a situation is described by the basic partial differential equations given by Chandrasekhar (1961), assuming a Boussinesq flow. A number of natural parameters emerge from the non-dimensionalization of these equations for the Rayleigh–Benard problem:

$$R = g\alpha d^3 \Delta T/\kappa\nu, \text{ the Rayleigh number,}$$

where ν is the kinematic viscosity, and κ the thermal diffusivity;

$$Q = \mu^* H_0^2\, d^2/4\pi\mu\eta, \text{ the Chandrasekhar number,}$$

$$Ta = 4\Omega_0^2 d^4/\nu^2, \text{ the Taylor number,}$$

$$\sigma = \nu/\kappa, \text{ the Prandtl number,}$$

and

$$\tau = \eta/\kappa, \text{ the magnetic Prandtl number.}$$

After linearization a normal mode analysis may be performed whereby the perturbations, which depend on time as well as the three spacial coordinates, are analysed into two-dimensional periodic waves. For example, the vertical component of velocity $w(x, y, z, t)$ has the form

$$w = W(z)\exp(\mathrm{i}(k_x + k_y) + pt),$$

where p, the frequency of the perturbation, may be complex and $k^2 = k_x^2 + k_y^2$ is the square of the wave number. This procedure leads to a separation of variables, where the horizonal dependence of the system is described by a function $f(x, y)$, which is a solution of the two-dimensional Helmholtz equation (Chandrasekhar 1961; Christopherson 1940).

The resulting vertical dependence of the disturbances are given by the equations (Chandrasekhar 1961)

$$(\mathrm{D}^2 - a^2 - P_1 n) F = -Ra^2 W, \tag{1}$$

$$(\mathrm{D}^2 - a^2 - P_2 n) H = -(H_0 d/\eta)\,\mathrm{D}W, \tag{2}$$

$$(\mathrm{D}^2 - a^2 - P_2 n)\chi = -(H_0 d/\eta)\,\mathrm{D}Z, \tag{3}$$

$$(\mathrm{D}^2 - a^2 - n) Z = -(2\Omega_0 d/\nu)\,\mathrm{D}W - (\mu^* H_0 d/4\pi\rho\nu)\,\mathrm{D}\chi, \tag{4}$$

$$(\mathrm{D}^2 - a^2)(\mathrm{D}^2 - a^2 - n) W + (\mu^* H_0 d/4\pi\rho\nu)\,\mathrm{D}(\mathrm{D}^2 - a^2) H - (2\Omega_0 d^3/\nu)\,\mathrm{D}Z = F, \tag{5}$$

where $n = pd^2/\nu$ is a non-dimensional time constant which is related to the gyration frequency by (Chandrasekhar 1961) $p/\Omega_0 = 2n/\sqrt{Ta}$, $a = kd$ is the aspect ratio of the cell, $P_1 = \nu/\kappa$, $P_2 = \nu/\eta$, $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}z$, $(0 \leqslant z \leqslant 1)$. The variables $W(z)$, $Z(z)$, $F(z)$, $H(z)$, $\chi(z)$ correspond to the vertical components of velocity and vorticity, temperature fluctuation, perturbed magnetic field and current density, respectively.

3. VARIATIONAL METHOD (RIGID–RIGID BOUNDARIES)

The critical Rayleigh number and cell size at the onset of oscillatory instability are determined by using a variational method, the validity of which has been established by Chandrasekhar (1961), and is an extension of that used by Murphy & Steiner (1975) for the stationary convection case relevant to the experiments of Nakagawa (1957).

The set of equations (1)–(5) can be combined to give one double eigenvalue equation. From (3) and (4)

$$[(\mathrm{D}^2 - a^2 - P_2 n)(\mathrm{D}^2 - a^2 - n) - Q\mathrm{D}^2] Z = -(2\Omega_0 d/\nu)(\mathrm{D}^2 - a^2 - P_2 n)\,\mathrm{D}W, \tag{6}$$

and from (2) and (5)

$$\begin{aligned}(\mathrm{D}^2 - a^2)[(D^2 - a^2 - P_2 n)(\mathrm{D}^2 - a^2 - n) - Q\mathrm{D}^2]^2 W + Ta(\mathrm{D}^2 - a^2 - P_2 n)^2 \mathrm{D}^2 W \\ = (\mathrm{D}^2 - a^2 - P_2 n)[(\mathrm{D}^2 - a^2 - P_2 n)(\mathrm{D}^2 - a^2 - n) - Q\mathrm{D}^2] F.\end{aligned} \tag{7}$$

In Nakagawa's (1957) experiment the fluid was held in a pyrex cylinder, the bottom of which was a stainless steel plate, and on the top surface of the mercury an oxidized film formed. The boundaries can then be idealized to rigid perfect conductors, as well as being non-deformable and isothermal. Hence at the boundaries, we have

$$W = \mathrm{D}W = F = Z = 0.$$

In an examination of the steady-state system, Murphy & Steiner (1975) employed the non-conducting boundary conditions, with some mention of the perfectly conducting boundaries. Although, as pointed out by the authors, the convective stability of the steady-state system is independent of the form of the magnetic field caused by a decoupling of the equations, it is dependent on the form of the current density, and hence it is important to accurately fix the magnetic–electric boundary conditions. As a first approximation, the boundaries are taken to be either perfect conductors, or non-conductors. Even though Nakagawa's (1957) experiments corresponded with perfectly conducting boundaries, both cases are considered here. This is done, not only for the sake of completeness, but also as a direct comparison of the Murphy & Steiner (1975) results. These two situations correspond to $H = 0$ and $\mathrm{D}\chi = 0$ at the boundaries when they are perfect conductors, and if they are non-conducting $\chi = 0$ and $\mathrm{D}H - aH = 0$ at the bottom boundary and $\chi = 0$ and $\mathrm{D}H + aH = 0$ at the top boundary.

Because of the symmetry of the problem, it is convenient to consider the layer to be bounded at $z = -\frac{1}{2}$ and $z = \frac{1}{2}$.

$F(z)$ is now expanded in a cosine series

$$F(z) = \sum_{m=0}^{J} A_m \cos(2m+1)\pi z, \tag{8}$$

which, in this form, satisfies the boundary conditions, and because of the linearity of the system, $W(z)$, $Z(z)$, $H(z)$ and $\chi(z)$ may be expressed as

$$\left.\begin{aligned} W(z) &= \sum_{m=0}^{J} A_m W_m(z), \\ Z(z) &= \sum_{m=0}^{J} A_m Z_m(z), \\ H(z) &= \sum_{m=0}^{J} A_m H_m(z), \\ \text{and} \qquad \chi(z) &= \sum_{m=0}^{J} A_m \chi_m(z). \end{aligned}\right\} \tag{9}$$

Substitution of the expansions (8) and (9) into (7) leads to

$$\begin{aligned} &\{(\mathrm{D}^2 - a^2)[(\mathrm{D}^2 - a^2 - P_2 n)(\mathrm{D}^2 - a^2 - n) - Q\mathrm{D}^2]^2 \\ &\qquad + Ta(\mathrm{D}^2 - a^2 - P_2 n)^2 \mathrm{D}^2\}\, W_m(z) = C_{2m+1} \cos(2m+1)\pi z, \end{aligned} \tag{10}$$

where

$$\begin{aligned} C_{2m+1} = &-[(2m+1)^2\pi^2 + a^2 + P_2 n] \\ &\times [((2m+1)^2\pi^2 + a^2 + P_2 n)((2m+1)^2\pi^2 + a^2 + n) + Q(2m+1)^2\pi^2], \\ &\qquad (m = 0, 1, 2, \ldots). \end{aligned} \tag{11}$$

The general solution of (10) is

$$W_m(z) = \sum_{j=1}^{5} B_j^{(m)} \cosh q_j z + C_{2m+1}\gamma_{2m+1} \cos(2m+1)\pi z, \tag{12}$$

with

$$\gamma_{2m+1}^{-1} = -(2m+1)^{10}\pi^{10} + a_1(2m+1)^8\pi^8 - a_2(2m+1)^6\pi^6 \\ + a_3(2m+1)^4\pi^4 - a_4(2m+1)^2\pi^2 + a_5,$$

where the $B_j^{(m)}$, $j = 1, 2, 3, 4, 5$ and $m = 0, 1, 2, \ldots$ are constants of integration and the $q_j^2 = y_j$ are the roots of quintic polynomial

$$(y-a^2)[(y-a^2-P_2 n)(y-a^2-n)-Qy]^2 + Tay(y-a^2-P_2 n)^2 = 0. \tag{13}$$

Introducing the adopted expansion for $Z(z)$ and the general solution for $W(z)$ now into (6) determines a non-homogeneous differential equation for $Z(z)$, which has a solution given by

$$Z_m(z) = -\left(\frac{2\Omega_0 d}{\nu}\right)\left[\sum_{j=1}^{5} \frac{\{(q_j^2-a^2-P_2\sigma)\, q_j B_j^{(m)} \sinh q_j z\}}{\{(q_j^2-a^2-P_2 n)(q_j^2-a^2-n)-Qq_j^2\}} \\ -\gamma_{2m+1}(2m+1)\pi[(2m+1)^2\pi^2+a^2+P_2 n]^2 \sin(2m+1)\pi z\right]. \tag{14}$$

The corresponding solution for $\chi_m(z)$ is

$$\chi_m(z) = \left(\frac{H_0 d}{\eta}\right)\left(\frac{2\Omega_0 d}{\nu}\right)\left[\sum_{j=1}^{5} \frac{q_j^2 B_j^{(m)} \cosh q_j z}{\{(q_j^2-a^2-P_2 n)(q_j^2-a_2-n)-Qq_j^2\}} \\ +\gamma_{2m+1}(2m+1)^2\pi^2[(2m+1)^2\pi^2+a^2+P_2 n] \cos(2m+1)\pi z\right], \tag{15}$$

which follows from (3).

Substituting for $H(z)$ and $W(z)$ into (2) leads to the following solution for $H_m(z)$

$$H_m(z) = -\left(\frac{H_0 d}{\eta}\right)\left[\sum_{j=1}^{5} \frac{q_j B_j^{(m)} \sinh q_j z}{\{q_j^2-a^2-P_2 n\}} - \frac{C_{2m+1}\gamma_{2m+1}(2m+1)\pi \sin(2m+1)\pi z}{\{(2m+1)^2\pi^2+a^2+P_2\sigma\}}\right]. \tag{16}$$

The constants $B_j^{(m)}$ are determined by the boundary conditions. When $W_m = 0$,

$$\sum_{j=1}^{5} B_j^{(m)} \cosh(\tfrac{1}{2}q_j) = 0, \tag{17}$$

when $DW_m = 0$,

$$\sum_{j=1}^{5} B_j^{(m)} q_j \sinh(\tfrac{1}{2}q_j) = (-1)^m C_{2m+1}\gamma_{2m+1}(2m+1)\pi, \tag{18}$$

when $Z_m = 0$,

$$\sum_{j=1}^{5} \frac{B_j^{(m)} q_j(q_j^2-a^2-P_2 n) \sinh(\tfrac{1}{2}q_j)}{\{(q_j^2-a^2-P_2 n)(q_j^2-a^2-n)-Qq_j^2\}} \\ = (-1)^m\gamma_{2m+1}(2m+1)\pi[(2m+1)^2\pi^2+a^2+P_2\sigma]^2. \tag{19}$$

If the boundaries are perfect conductors, $H_m = 0$, and

$$\sum_{j=1}^{5} \frac{B_j^{(m)} q_j \sinh(\tfrac{1}{2}q_j)}{\{q_j^2 - a^2 - P_2 n\}} = \frac{(-1)^{m+1} C_{2m+1} \gamma_{2m+1} (2m+1)\pi}{\{(2m+1)^2\pi^2 + a^2 + P_2 n\}}, \tag{20}$$

with $D\chi_m = 0$,

$$\sum_{j=1}^{5} \frac{B_j^{(m)} q_j^3 \sinh(\tfrac{1}{2}q_j)}{\{(q_j^2 - a^2 - P_2 \sigma)(q_j^2 - a^2 - \sigma) - Q q_j^2\}} = (-1)^m \gamma_{2m+1} (2m+1)^3 \pi^3 [(2m+1)^2\pi^2 + a^2 + P_2 n]. \tag{21}$$

For non-conducting boundaries $\mathrm{D}H \pm aH = 0$ leads to

$$\sum_{j=1}^{5} \frac{B_j^{(m)} q_j [q_j \cosh(\tfrac{1}{2}q_j) + a \sinh(\tfrac{1}{2}q_j)]}{\{q_j^2 - a^2 - P_2 n\}} = \frac{(-1)^{m+1} C_{2m+1} \gamma_{2m+1} a (2m+1)\pi}{[(2m+1)^2\pi^2 + a^2 + P_2 n]}, \tag{22}$$

and because $\chi_m = 0$,

$$\sum_{j=1}^{5} \frac{B_j^{(m)} q_j^2 \cosh(\tfrac{1}{2}q_j)}{\{(q_j^2 - a^2 - P_2 n)(q_j^2 - a^2 - n) - Q q_j^2\}} = 0. \tag{23}$$

Substituting the expansions for $F(z)$ and $W(z)$ into (1), multiplying both sides by $\cos(2l+1)\pi z$, $l = 0, 1, 2, \ldots$, and integrating with respect to z, $-\frac{1}{2} \leqslant z \leqslant \frac{1}{2}$, yields

$$\begin{aligned} A_l\{(2l+1)^2\pi^2 + a^2 + P_1 n\} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos^2(2l+1)\pi z \,\mathrm{d}z &= Ra^2 \Bigg\{ A_l C_{2l+1} \gamma_{2l+1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos^2(2l+1)\pi z \,\mathrm{d}z \\ &\quad + \sum_{m=0}^{J} A_m \sum_{j=1}^{5} B_j^{(m)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cosh(q_j z) \cos(2l+1)\pi z \,\mathrm{d}z \Bigg\}. \end{aligned} \tag{24}$$

Using Chandrasekhar's (1961) notation, let

$$\begin{aligned} \left(\frac{l}{m}\right) &= \sum_{j=1}^{5} B_j^{(m)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cosh(q_j z) \cos(2l+1)\pi z \,\mathrm{d}z \\ &= (-1)^l 2(2l+1)\pi \sum_{j=1}^{5} \frac{B_j^{(m)} \cosh(\tfrac{1}{2}q_j)}{\{q_j^2 + (2l+1)^2\pi^2\}}. \end{aligned} \tag{25}$$

Equation (24) is a linear homogeneous system for the determination of the A_m, which will have a solution if

$$\left\| \frac{1}{2} \left[\frac{\{(2l+1)^2\pi^2 + a^2 + P_1 n\}}{Ra^2} - C_{2l+1} \gamma_{2l+1} \right] \delta_{lm} - \left(\frac{l}{m}\right) \right\| = 0, \quad l, m = 0, 1, 2, \ldots, J. \tag{26}$$

The characteristic equation (26) is a complex double eigenvalue problem in R and n. Physically, R, the Rayleigh number, is a real physical number, whereas n in general is complex. At a stationary bifurcation, $n = 0$ and (26) reduces to a single eigenvalue equation. This was the case treated in Murphy & Steiner (1975).

Whereas for the onset of oscillatory instabilities, n is a purely imaginary number, i.e. $\mathrm{Re}(n) = 0$.

4. NUMERICAL METHOD

The critical values of the Rayleigh number at which oscillatory instability sets in are now determined and compared to the experimental values of Nakagawa (1957). The corresponding critical cell sizes and gyration frequencies are also established. This is achieved by iterating in (26) with respect to n, so that the imaginary part of R vanishes. The numerical method treats the Rayleigh number as being complex, and the physics of the system requires that it be real. The relevant values of Q, Ta, P_1 and P_2 corresponding to Nakagawa's (1957) experiment are used, i.e. $Ta = 7.75\times 10^5\,\pi^4$, $P_1 = 0.025$, $P_2 = 1.5\times 10^{-7}$, and Q varies from π^2 to *ca.* $10^{3.5}\,\pi^2$. A half interval search over a is employed to find R_c, the minimum R for variable a, and this value of a is then termed the critical cell size a_c. The solutions can be 'fine-tuned' by including successive higher approximations in the secular determinant (26), achieved by increasing J. However, it was found that the second approximation generally gave the desired accuracy.

To proceed in this manner, the complex quantities q_j and $B_j^{(m)}$, $j = 1,2,3,4,5$ and $m = 0,1,2,\ldots,J$ need to be evaluated. The values of q_j^2 correspond to the roots of the quintic polynomial equation (13) which has complex coefficients; recalling that the frequency for any oscillatory motion is always imaginary. A numerical procedure due to Jenkins & Traub (1972) was employed to find these complex roots.

Next the $B_j^{(m)}$ are evaluated by using (17), (18) and (19) together with (20) and (21) when the boundaries are perfect conductors, or (22) and (23) when they are non-conducting. These equations are split into real and imaginary parts, resulting in a system of ten equations which are solved by using a Gaussian elimination procedure with pivoting.

The numerical procedure was tested by solving for R_c with the non-conducting boundary condition for stationary convection and the results match those of Murphy & Steiner (1975) precisely, as can be seen by comparing our figure 1*a* with their figure 7.

5. RESULTS

If the onset of instability is via an oscillatory bifurcation, the diffusivities of the fluid play an important role in determining the precise nature of the instability curve. In contrast, the stationary bifurcation is explicitly independent of the relative values of the diffusivities, as gauged by the Prandtl numbers. Further, if $\kappa > \eta$ stability can be broken by the excitation of oscillations of increasing amplitude when a magnetic field acts alone and if $\nu > \kappa$ an oscillatory bifurcation is not possible when rotation is the only external constraint on the system. When rotation and a magnetic field act in conjunction the type of instability can only be established from a numerical investigation over parameter space.

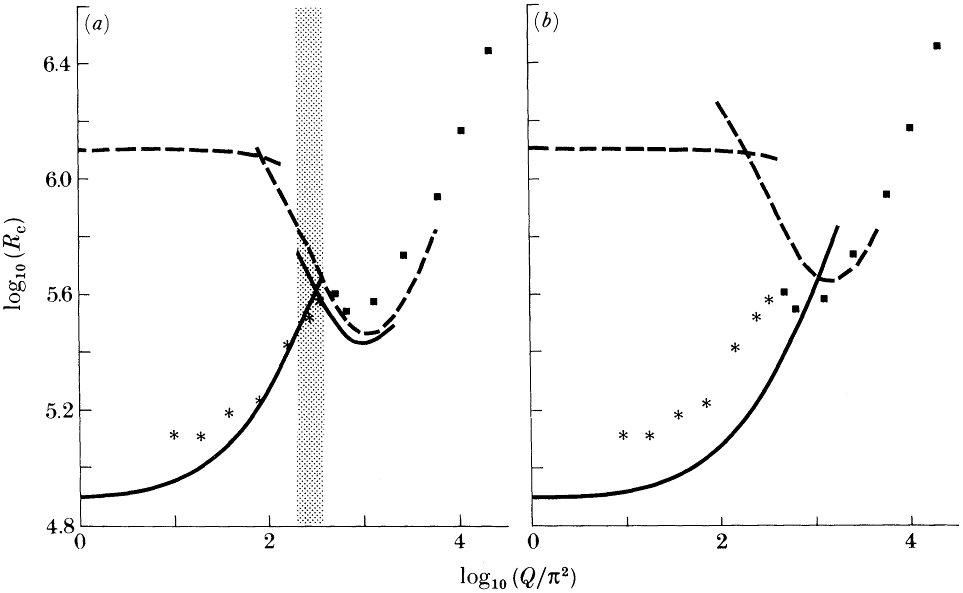

FIGURE 1. Variation of R_c with the magnetic field strength Q when $Ta = 7.75 \times 10^5\pi^4$, $P_1 = 0.025$, and $P_2 = 1.5 \times 10^{-7}$ in the case of top and bottom rigid conducting boundaries (*a*), and in the case of top and bottom rigid non-conducting boundaries (*b*). The shaded region represents the range of Q values where the oscillatory instability R–a curve has two minima. Full and broken curves are for oscillatory and stationary instability respectively. Nakagawa's (1957) experimental results are: *, oscillatory instability and ■, stationary instability.

Both the stationary instability and the oscillatory instability curves were established for the perfectly conducting and non-conducting boundary conditions. The direct instability curve for the non-conducting boundaries also follows from the numerical model of Murphy & Steiner (1975), the results from which agree precisely with those presented here. However, the results for the perfectly conducting boundaries, which describe the experimental situation (Nakagawa 1957) more closely, do in fact fit the experimental data better. These features of the results may be seen from figure 1*a*, *b*.

At low values of the Chandrasekhar number it is apparent that the stationary instability curves are independent of the magnetic–electric boundary conditions and not until the second minimum in the R–a relationship is present do these boundary conditions play any discriminating role in the determination of R_c. The range over Q in which the R–a relation has two minima is considerably smaller for the perfectly conducting stationary instability case.

It has been demonstrated numerically that for $Q \lesssim 10^3\pi^2$, the critical value of the Rayleigh number is lower for an oscillatory bifurcation than it is for a stationary bifurcation, hence the onset of instability is oscillatory. For Q between *ca.* $10^3\pi^2$ and *ca.* $10^{3.25}\pi^2$, an oscillatory bifurcation is still possible, but in this approximate region of Q, R_c for stationary bifurcation is lower in the case of non-conducting boundaries, resulting in the system becoming unstable to over-

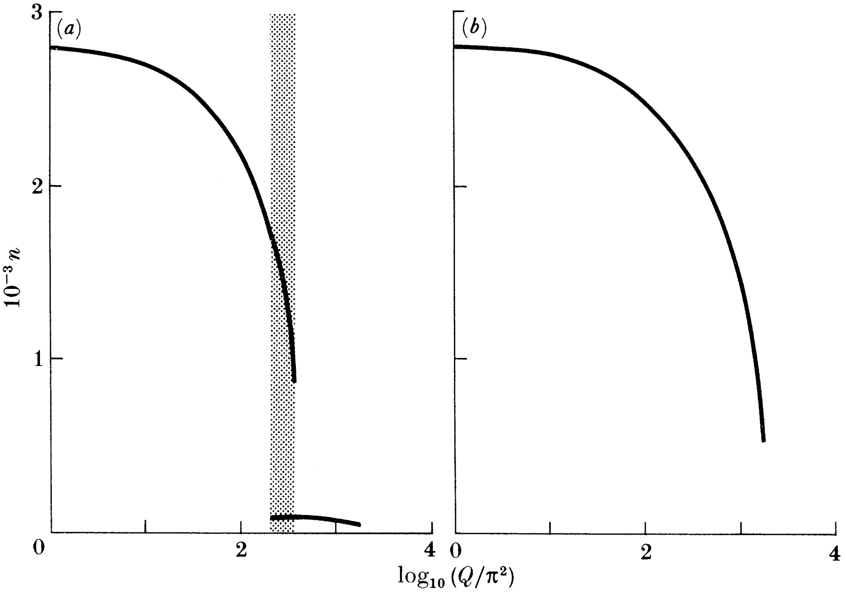

FIGURE 2. Variation of n ($2n/\sqrt{Ta} = p/\Omega_0$, the gyration frequency) with the magnetic field strength Q when $Ta = 7.75\times10^5\pi^4$, $P_1 = 0.025$ and $P_2 = 1.5\times10^{-7}$ in the case of top and bottom rigid conducting boundaries (*a*), and in the case of top and bottom rigid non-conducting boundaries (*b*). The shaded region represents the range of Q values where the oscillatory instability R–a curve has two minima.

TABLE 1. THE DEPENDENCE OF THE RAYLEIGH NUMBER AT THE ONSET OF OSCILLATORY INSTABILITY, AND CORRESPONDING WAVE NUMBER, ON THE MAGNETIC FIELD STRENGTH WHEN $Ta = 7.75\,\pi^4\times10^5$, $P_1 = 0.025$ AND $P_2 = 1.5\times10^{-7}$ FOR TWO RIGID CONDUCTING BOUNDARIES

(Secondary minima on the R–a curve are shown in brackets.)

$Q\times10^{-3}$	$a_{\min}$		$R_{\min}\times10^{-5}$		b/cm
0.1	8.95		0.911		2.43
1.0	10.12		1.883		2.15
2.0	10.83	(4.99)	2.950	(5.807)	2.01
3.0	11.40	(4.99)	4.085	(4.265)	1.91
3.5	(11.59)	4.99	(4.656)	3.888	4.36
4.0		5.10		3.583	4.26
5.0		5.31		3.176	4.10
6.0		5.48		2.947	3.97
10.0		6.22		2.677	3.49
17.5		7.42		3.045	2.93

turning convective motions for a weaker buoyancy force than is needed for oscillatory motions.

For Q beyond *ca.* $10^{3.25}\pi^2$, oscillatory bifurcations are no longer possible. The frequency of the oscillations, as shown in figure 2*a*, *b*, rapidly drops off to zero as Q is increased beyond *ca.* $10^2\pi^2$. However, contrary to intuitive expectations, the two instability curves do not coalesce.

In common with the stationary cases, the R–a relationship for oscillatory

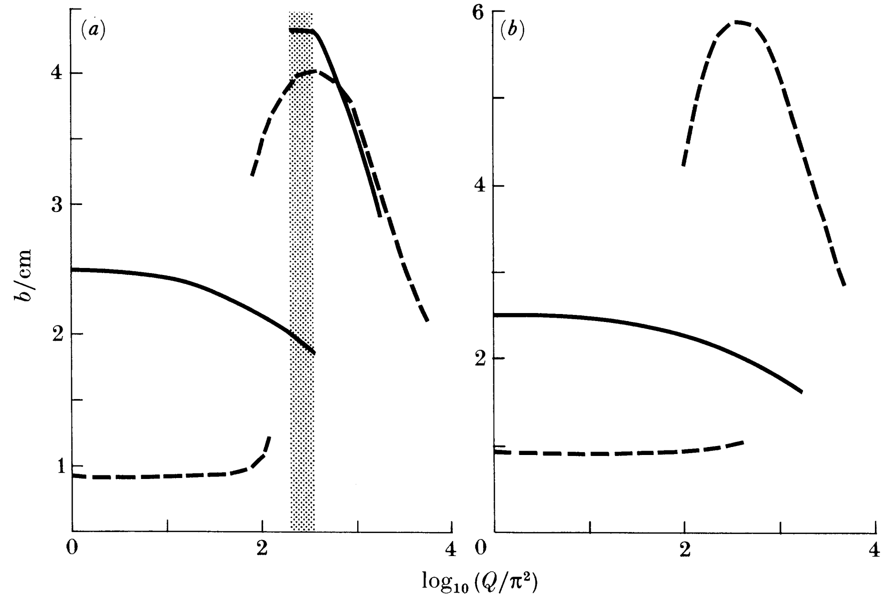

FIGURE 3. Variation of the convective cell size b with the magnetic field strength Q when $Ta = 7.75 \times 10^5\pi^4$, $P_1 = 0.025$ and $P_2 = 1.5 \times 10^{-7}$ in the case of top and bottom rigid conducting boundaries (a), and in the case of top and bottom rigid non-conducting boundaries (b). The shaded region represents the range of Q values where the oscillatory instability R–a curve has two minima. Full and broken curves are for the onset of oscillatory instability and stationary instability respectively.

bifurcations with perfectly conducting boundaries also has two minima in a certain range of Q, which is designated by the equivalent shaded regions in figures 1a, 2a and 3a. Actual values for a_c and R_c are given by the entries in table 1 where the secondary minimum values are enclosed in brackets and the discontinuous change in wave number has been indicated by an arrow. For the parameters considered, the non-conducting case does not exhibit the double minimum on the R–a curve and hence one would expect a sudden transition from oscillatory behaviour to steady behaviour as Q is increased, accompanied by a sudden widening of the cell. The cell width is measured by a parameter $b = 4\pi d/a\sqrt{3}$ (Chandrasekhar 1961), b being the distance between the centres of two adjacent hexagonals, a is the aspect ratio of the cell as defined earlier, and d is the depth of the layer, which in Nakagawa's (1957) experiment was 3 cm. The variation of cell width with Q is given in figure 3a, b for each of the boundary conditions.

In the case when the perfectly conducting boundary conditions apply, the oscillatory instability curve has two branches and one would expect to observe a sudden transition in the oscillatory behaviour as Q is increased, followed at even higher Q by a further transition from oscillatory to steady behaviour. However, Nakagawa's (1957) experiments only describe one transition, that being from oscillatory to steady. From figure 1a, it is found that the value of Q at which the experimental transition takes place corresponds with the value of Q at which the oscillatory branches cross. From figure 2a, at that point in Q, the oscillations undergo a sudden (discontinuous) decrease in frequency. The frequency of this second branch of the oscillatory instability curve may be so low that experimentally, it has been taken to be zero, and consequently classified as a stationary bifurcation. This branch, while having a lower R_c than the stationary equivalent, follows it very closely before terminating at a value of the Chandrasekhar number which is approximately equal to $10^{3.25}\pi^2$. The cell sizes at the onset of instability via a stationary bifurcation are also very close to those corresponding to the second oscillatory branch as seen in figure 3*a*, under these circumstances it would be difficult, experimentally, to distinguish between the two cases.

A feature of particular interest concerning the perfectly conducting case is that for a range of $Q(10^{2.5}\pi^2 \lesssim Q \lesssim 10^3\pi^2)$, an increase in the strength of the impressed magnetic field actually enhances the onset of convection, as can readily be seen from figure 1*a* where there is a drop in R_c as Q increased through this range.

Unfortunately, owing to the layer of mercury's forming an oxidized film on the top, Nakagawa (1957) was unable to measure the cell sizes at the onset of instability and hence observe the discontinuous transition in cell sizes as Q is increased.

6. CONCLUSION

Chandrasekhar's (1961) determination of the parametric dependence of the onset of convective instability in a rotating layer of mercury subject to a vertical magnetic field, in the particular case when the free–free boundary conditions apply, exhibited all the significant features observed in Nakagawa's (1957, 1959) experiments. Even when the terms involving the ratio of the Prandtl number to the magnetic Prandtl number, which is small in the case of mercury, were suppressed the evident discontinuities in cell sizes at the onset of instability with increasing magnetic field strength at constant speeds of rotation were qualitatively established as well as the nature of the instability.

In this study, where the adoption of the theoretical rigid–rigid boundary conditions gives a better representation of Nakagawa's experimental arrangement, our computations for the critical Rayleigh number at the onset of oscillatory instability for assigned values of the Chandrasekhar number, Taylor number, Prandtl number, magnetic Prandtl number and horizontal wave number are based on, what appears to be, the first explicit formulation of the variational principle utilizing these boundary conditions, as postulated by Chandrasekhar (1961). The theoretical oscillatory instability curve obtained for R_c as a function of magnetic field strength, with the parameter values used in the experiments, demonstrates that the electrical properties of the top and bottom boundaries are also significant, with good quantitative agreement being established for conducting boundaries, which were part of the experimental set up. Specifically, these results also show that in the case of non-conducting boundaries only one branch of this oscillatory instability curve exists and consequently no discontinuities in gyration frequency or cell size would be observed experimentally, provided the mode of instability did not change to steady cellular convection. Experimental investigations to determine the convective instability characteristics of fluids with different Prandtl numbers and subject to the same external constraints, do not seem to have been undertaken following the pioneering work of Nakagawa.

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