

Two-Scale Contact Process and Effects of Habitat Fragmentation on Metapopulations

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Received June 12, 2006, revised March 18, 2008

Abstract. Particle systems are usually defined on a homogeneous graph, the interaction neighborhoods of sites of the graph are linked to each other by translation. In this article, we study the contact process on a non homogeneous graph designed to model the interactions within metapopulations. The d -dimensional lattice is turned into a “chessboard” through the superposition of a mesoscopic lattice on the usual microscopic lattice. Each site of the mesoscopic lattice is the center of a square of the chessboard. Interactions occur at both site (microscopic) level and square (mesoscopic) level. The superposition of two interaction levels induces two birth rates, called microscopic and mesoscopic birth rates. Similarly, deaths occur at both levels: individual deaths at microscopic level, and mass extinctions (destruction of all the particles contained in a given square) at mesoscopic level. Our “two-scale” contact process can be viewed as a metapopulation model describing the evolution of a set of interacting local populations. We study the effect of coarseness, defined as the ratio of the mesoscopic scale over the microscopic scale, on the survival probability of the particle system. We find that, in the absence of mass extinctions, particles are more likely to spread out as coarseness increases, even if the mesoscopic birth rate decreases significantly with the square size. In the presence of mass extinctions, coarseness has only a limited effect.

KEYWORDS: interacting particle system, contact process on a finite set, spatial structure, multiscale argument, spatial ecology, metapopulation

AMS SUBJECT CLASSIFICATION: Primary 60K35, 82C22, Secondary 92D25, 92D40

1. Introduction

The contact process is a stochastic model including a spatial structure in the form of local interactions [12]. Each site of the d -dimensional integer lattice is either empty or occupied by a particle. Empty sites become occupied at a rate proportional to the number of particles present in some interaction neighborhood. The proportionality constant is usually called the birth rate of the particles. Occupied sites become empty at rate 1, regardless of the state of their neighbors. In any dimension, it is known that the system exhibits a phase transition, that is, starting from the “all occupied” configuration, each site of the lattice gets occupied infinitely many often if and only if the birth rate exceeds some threshold [15]. The smaller the dimension, the greater the threshold for the particles to spread.

The contact process is ideally suited to investigate the evolution of spatially homogeneous populations. Many populations, however, live in patchy habitats due to habitat fragmentation or naturally occurring spatial heterogeneities [16]. Such sets of interacting local populations are commonly called metapopulations (see [10] for more details about the ecological concept of metapopulation). In this article, we investigate an extension of the contact process with interactions at two scale levels in order to model the evolution of such metapopulations. Our process can be seen as a contact type process evolving on some inhomogeneous graph. We think of the integer lattice as an infinite “chessboard”, each square of the chessboard containing the same number of sites, and assume that interactions occur at both site level (or microscopic level) and square level (or mesoscopic level). The superposition of microscopic and mesoscopic lattices induces two birth rates, namely a microscopic birth rate (the rate at which a particle gives birth onto adjacent sites of the same square), and a mesoscopic birth rate (the rate at which a particle gives birth onto adjacent squares), also called migration rate. In the same way, a specific death rate is associated to each lattice: an individual death rate at microscopic level, and a mass extinction rate (destruction of all the particles contained in the same square) at mesoscopic level. Our first results show that, similarly to the basic contact process, our “two-scale” contact process exhibits phase transitions with respect to each of its birth rates. That is, for each fixed microscopic (respectively, mesoscopic) birth rate, survival may occur if and only if the mesoscopic birth rate (respectively, the microscopic birth rate) exceeds some critical value. In order to obtain a lower bound for the critical birth rates, we also give an explicit condition for the process to go extinct. Another objective is to determine the effect of the square size on the interacting particle system. We find that, in the absence of mass extinctions, particles are more likely to spread out as the square size increases (even if the mesoscopic birth rate decreases significantly with the square size), while, in the presence of mass extinctions, the square size has only a limited effect.

To define rigorously our spatial structure, we let $N \geq 1$ be an integer referred to as the space scale, and denote by $L_1 = \mathbb{Z}^d$ and $L_N = (2N - 1)\mathbb{Z}^d$ the microscopic lattice and the mesoscopic lattice, respectively. We tile L_1 with squares of length side $2N - 1$ by setting

$$\mathcal{H} = (-N, N)^d \quad \text{and} \quad \mathcal{H}_z = (2N - 1)z + \mathcal{H} \quad \text{for any } z \in \mathbb{Z}^d.$$

In particular, we have the following one-to-one correspondence between squares and sites of the mesoscopic lattice: for any $z \in \mathbb{Z}^d$, there exists a unique $x \in L_N$ such that $\mathcal{H}_z = x + \mathcal{H}$, with x denoting the center of \mathcal{H}_z . We shall use the 2-dimensional terminology like “chessboard” or “square” even if our results hold in any dimension. Let \asymp define the equivalence relation

$$\forall x, y \in L_1, \quad x \asymp y \quad \text{if and only if} \quad \exists z \in \mathbb{Z}^d \quad \text{such that} \quad x, y \in \mathcal{H}_z.$$

To describe the contacts at the microscopic level (between adjacent sites), we introduce an interaction neighborhood: $x \smile y$ indicates that y is one of the $2d$ nearest neighbors of x , i.e.,

$$\forall x, y \in L_1, \quad x \smile y \quad \text{if and only if} \quad \|x - y\|_1 = 1$$

where $\|x - y\|_1 = |x_1 - y_1| + \dots + |x_d - y_d|$. To take into account the contacts at the mesoscopic level (between the centers of adjacent squares), we introduce another interaction neighborhood by defining the following binary relation between centers of adjacent squares:

$$\forall x, y \in L_N, \quad x \frown y \quad \text{if and only if} \quad \|x - y\|_1 = 2N - 1.$$

The two-scale contact process is a continuous-time Markov process in which the state at time t is a function $\eta_t : \mathbb{Z}^d \rightarrow \{0, 1\}$. A site $x \in \mathbb{Z}^d$ is said to be empty if $\eta_t(x) = 0$, and occupied by a particle otherwise. To describe the evolution rules of the process, we set, for $i = 0, 1$,

$$\eta^{x,i}(z) = \begin{cases} i, & \text{if } z = x, \\ \eta(z), & \text{otherwise} \end{cases} \quad \text{and} \quad \eta^x(z) = \begin{cases} 0, & \text{if } z \asymp x, \\ \eta(z), & \text{otherwise.} \end{cases}$$

In other words, $\eta^{x,0}$ (respectively, $\eta^{x,1}$) is the configuration obtained from η by removing the particle at site x if it exists (respectively, by adding a particle at site x if the site is not already occupied). The configuration η^x is obtained from η by removing all the particles from the square containing site x . The evolution rules are formally given by the pregenerator $\Omega = \Omega_{mic} + \Omega_{mes}$ where, for any cylinder function f of the configuration η ,

$$\Omega_{mic}f(\eta) = \beta \sum_{x \in L_1} \sum_{z \smile x} \eta(z) \mathbb{1}\{z \asymp x\} [f(\eta^{x,1}) - f(\eta)] + \delta \sum_{x \in L_1} [f(\eta^{x,0}) - f(\eta)]$$

describes the microscopic interactions and where

$$\Omega_{mes}f(\eta) = B \sum_{x \in L_N} \sum_{z \sim x} \eta(z)[f(\eta^{x,1}) - f(\eta)] + D \sum_{x \in L_N} [f(\eta^x) - f(\eta)]$$

describes the mesoscopic interactions. To have an intuitive understanding of the dynamics, it is convenient to consider the process as a version of the contact process evolving on a non homogeneous graph that we called metapopulation graph. The set of sites is \mathbb{Z}^d . Two adjacent sites belonging to the same square are connected with an edge of length 1, and the centers of two adjacent squares are connected with an edge of length $2N - 1$. Figure 1 shows a picture of the metapopulation graph when $d = 2$ and $N = 2$. Particles give birth through the edges of length 1 at rate β (microscopic interaction) and through the edges of length $2N - 1$ at rate B (mesoscopic interaction). In any case, births onto already occupied sites are suppressed. Particles, independently of each other, die at rate δ , while mass extinctions (all the particles in a given square die simultaneously) occur at rate D . See Figure 2 for pictures of the process. The parameters β and B will be called the microscopic birth rate and the mesoscopic birth rate, respectively. The parameters δ and D will be called the individual death rate and the mass extinction rate, respectively.

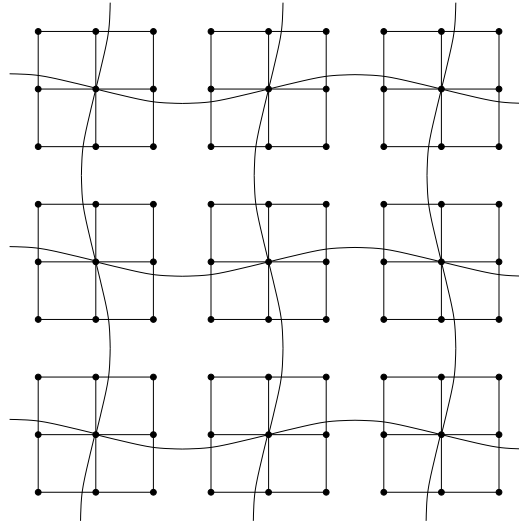
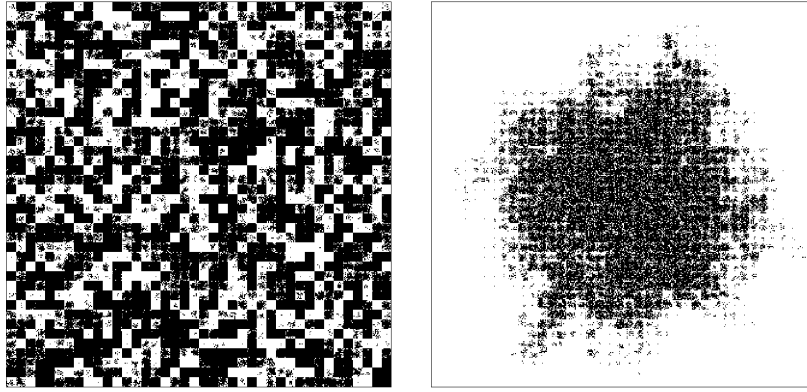


Figure 1. Metapopulation graph.

We first discuss the existence of a nontrivial stationary distribution (where nontrivial means different from the “all 0” configuration) depending on the birth rates. If we think of the process as being generated by a collection of



(a) $\beta = 15$, $B = 10$, $\delta = 0$ and $D = 1$ (b) $\beta = 8$, $B = 10$, $\delta = 1$ and $D = 0$

Figure 2. Snapshots of the two-scale contact process on the 400×400 square with periodic boundary conditions, starting with a single particle at site 0.

independent Poisson processes [11], a standard coupling argument allows us to prove that the process η_t is attractive and that

Lemma 1.1. *Let P_0^η denote the law of the process starting with a single particle at site 0. For all D and δ fixed, the survival probability $P_0^\eta(|\eta_t| \geq 1 \text{ for all } t \geq 0)$ is nondecreasing with respect to the birth rates B and β , and the scale parameter N .*

We now observe that, when $N = 1$, the value of β is irrelevant and the process reduces to a time change of the contact process with parameter $\lambda = B/(D + \delta)$. In this case, there exists a critical value $\lambda_c \in (0, \infty)$ such that the following holds: If $\lambda \leq \lambda_c$ the process converges in distribution to the “all 0” configuration, while if $\lambda > \lambda_c$ there is a nontrivial stationary distribution (see [15, Theorem 2.25] or [3]).

When $\beta = 0$ and $N \geq 2$, the particles concentrate on L_N and the process η_t viewed on the mesoscopic lattice becomes a contact process with parameter $\lambda = B/(D + \delta)$, which implies as previously that there exists a nontrivial stationary distribution if and only if $\lambda > \lambda_c$. Assuming that $N \geq 2$ and $\beta > 0$, survival depends on the combined effects of the birth rates β and B . More precisely,

Theorem 1.1. *Assume that $\delta > 0$. For all $N \geq 2$ and $\beta > 0$ there is a critical value $B_c \in (0, \infty)$ such that*

1. *If $B < B_c$, the process converges to the “all 0” configuration while.*
2. *If $B > B_c$, there is a nontrivial stationary distribution.*

Theorem 1.2. Assume that $\delta > 0$. For all $N \geq 2$ and $D\lambda_c < B < (D + \delta)\lambda_c$ there is $\beta_c \in (0, \infty)$ such that

1. If $\beta < \beta_c$, the process converges to the “all 0” configuration while.
2. If $\beta > \beta_c$, there is a nontrivial stationary distribution.

The proof essentially relies on coupling arguments, the aim is to compare the two-scale contact process with well-known particle systems. The comparison is basically obtained thanks to a “destruction” of the spatial structure of each square. In other words, the idea is to consider the process that describes

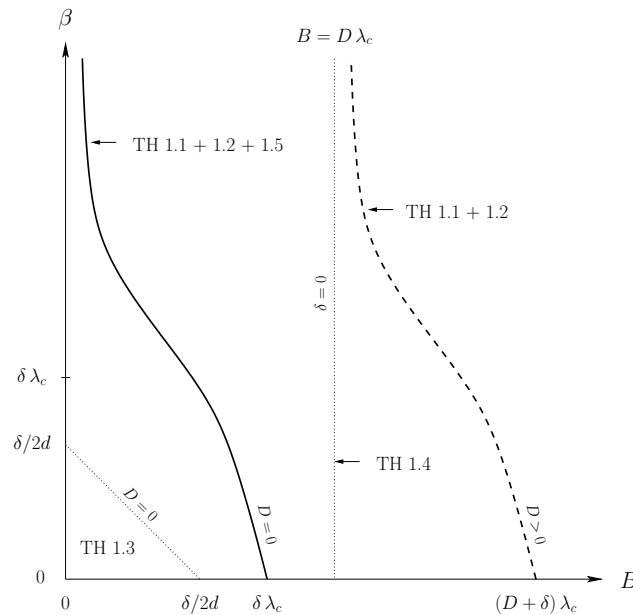


Figure 3. Phase diagram of the process. When $\delta = 0$, the limiting behavior of the process depends neither on β nor on N and the phase transition occurs along the straight line with equation $B = D\lambda_c$. When $\delta > 0$, the transition occurs along the continuous or the dashed curve depending on whether $D = 0$ or $D > 0$. Part 1 of Theorem 1.1 and Part 2 of Theorem 1.2 imply that the vertical axis is an asymptote of the continuous curve. We conjecture that the straight line with equation $B = D\lambda_c$ is an asymptote of the dashed curve. By Theorem 1.5, the continuous curve converges pointwise to the vertical axis for all $\beta > \delta\lambda_c$. By Theorem 1.3, both curves are bounded from below by the dotted segment on the bottom-left corner of the picture. By Lemma 1.1, the curves are nonincreasing.

the evolution at a mesoscopic level (see Section 3) by counting the number of particles in each square without taking into account their spatial configuration.

We now give an explicit lower bound on the critical values we have just introduced relying on techniques introduced in [6]. When $\beta = 0$, the process viewed on the mesoscopic lattice reduces to a time change of the basic contact process with parameter $\lambda = B/(D + \delta)$. Through a comparison with a branching random walk, it is straightforward to deduce that the process converges to the “all 0” configuration when $2dB < D + \delta$. When both β and B are different from 0, the long-term behavior of the process is more complicated to predict due to the combined effects of the two birth rates. We can however extend the result in the following way.

Theorem 1.3. *The process converges to the “all 0” configuration whenever $2d(B + \beta) < D + \delta$.*

Note that the condition in Theorem 1.3 is uniform in N .

The last step is to investigate the connection between the spatial parameter N and the survival probability of the particle system. In particular, we find that the effects of habitat fragmentation strongly depends on the death mechanism, that is mass extinction or individual deaths. First of all, note that Lemma 1.1 implies that, the birth and death rates being fixed, particles are more likely to spread out as N increases in the sense that

$$P_0^n(|\eta_t| \geq 1 \text{ for all } t \geq 0)$$

is nondecreasing with respect to N . In the context of metapopulations, migrations (births of particles sent to the center of adjacent squares) and mass extinctions are usually made harder as the size $(2N - 1)^d$ of the local populations increases. To take this factor into account, we now assume that the mesoscopic birth rate $B = B_N$ and the mass extinction rate $D = D_N$ are functions decreasing in N . Our next result shows that in the limiting case of pure mass extinctions the survival probability does not depend on N provided the ratio B_N/D_N is held constant. The result is essentially due to the fact that, in this case, the square size does not affect the extinction time of the process restricted to a single square. More precisely, we have the following

Theorem 1.4 (Mass extinction). *Let $\delta = 0$. Then there exists a nontrivial stationary distribution if and only if $B_N > D_N \lambda_c$ regardless of the values of β and N .*

In contrast, increasing N strongly promotes survival of the metapopulation in the absence of mass extinction, that is when $D_N = 0$, even if the mesoscopic birth rate B_N decreases exponentially fast with the square size, as stated in the following

Theorem 1.5 (Individual deaths). *Let $\delta = 1$ and $D_N = 0$. There exists a constant $b > 0$ such that, whenever $\beta > \lambda_c$ and $B_N > \exp(-bN^d)$, there is a nontrivial stationary distribution provided the space scale N is sufficiently large.*

The confrontation of Theorem 1.4 and Theorem 1.5 shows that the inclusion of mass extinctions reduces significantly the effects of the square size on the evolution of our two-scale model, with no correlation between N and the survival probability of the process with only mass extinctions, and a strong dependency between N and the survival probability of the process with only individual deaths. Interestingly, in the case when $\beta > \lambda_c$ and $B_N > \exp(-bN^d)$, mass extinctions occurring at a very low rate (for instance, $D_N = (2N - 1)^{-d}$) are much more devastating than individual deaths occurring at rate 1 when N is large. The effect of N on the long-term behavior of the metapopulation with no mass extinction appears clearly when the birth rate B does not depend on N . In this case, Lemma 1.1 and Theorem 1.5 imply the existence of a critical value $N_c \geq 1$ such that the metapopulation survives if and only if $N \geq N_c$. Moreover, the critical value N_c can be made arbitrarily large by taking $B_N \equiv B$ small, a straightforward consequence of Theorem 1.1. The basic idea behind the proof of Theorem 1.5 is the following. The extinction time of the process restricted to a single square of length side $2N - 1$ turns out to be exponential in N^d with probability close to 1 (see the study of the contact process on a finite set in [8] and [15]). When restoring the interactions on the mesoscopic lattice by setting $B_N > 0$, it can be proved that the time required to invade an empty square (surrounded by a large amount of particles) with a particle whose offspring will live an exponentially long time is smaller than a constant times $1/B_N$. In particular, even when $B_N = \exp(-bN^d)$, the invasion time can be made much shorter than the extinction time provided b is small and N is large. By dividing the time of the process into slices of height T_N and pretending that a cube in the resulting space-time partition is occupied if the center of the corresponding square is occupied by a particle a positive fraction of time, we find that, for a suitable T_N and all N sufficiently large, the set of occupied cubes percolate with positive probability. This fact that increasing the square size promotes survival has already been proved by Lanchier and Neuhauser [13, Theorem 3], for the contact process in a heterogeneous environment. Precisely, their result shows that the basic contact process with parameter $\lambda > \lambda_c$ evolving on the black squares of the infinite chessboard may survive provided the square size is large. Their proof, however, relies on different techniques, namely duality.

To investigate metapopulations, one might be interested in square-square interactions (all the particles in a given square can give birth to a particle which is sent to a randomly chosen site of an adjacent square) rather than site-site interactions on the mesoscopic lattice. In this case, we have the following

results. First of all, when $\beta = 0$, the number of particles in a given square jumps from i to $i + 1$ at rate $B((2N - 1)^d - i)$ times the number of particles present in the $2d$ adjacent squares, which implies that the metapopulation goes extinct whenever

$$2d(B(2N - 1)^d + \beta) < D + \delta.$$

This is a straightforward consequence of the proof of Theorem 1.3. Lemma 1.1 still holds. Since in addition the metapopulation is more likely to survive than in the two-scale contact process, we obtain the existence of a unique critical curve dividing the phase diagram into two regimes: survival and extinction. Finally, in contrast with the two-scale contact process, the parameter N has a strong effect on the survival of the metapopulation even in the limiting case of pure mass extinctions. The key of the proof is to compare the process viewed at the mesoscopic scale with the contact process with parameter $B(2N - 1)^d/(D + \delta)$. The comparison is done by considering the process modified so that particles sent to a square already occupied by a particle are killed. In particular, whenever $B > 0$, survival occurs for N large.

The rest of the article is devoted to proofs. In Section 2, we rely on coupling arguments to investigate the monotonicity of the process, and prove Lemma 1.1. In Section 3, we introduce a mesoscopic version of our process to deduce Theorems 1.1, 1.2 and 1.4, while the proof of Theorem 1.3 is carried out in Section 4. Finally, in Section 5, we construct the percolation structure described above, and prove Theorem 1.5.

2. Construction of the process. Proof of Lemma 1.1

This section is devoted to the proof of Lemma 1.1 whose first step is to construct the process from collections of independent Poisson processes, which is referred to as Harris' graphical representation [11]. For each $x, z \in L_1$ with $x \smile z$ and $x \asymp z$, let $\{T_n^\beta(x, z) : n \geq 1\}$ denote the arrival times of independent Poisson processes with rate β , and draw an arrow from site x to site z at time $T_n^\beta(x, z)$ to indicate that a birth may occur. To take into account the interactions at the mesoscopic level, we introduce, for any $x, z \in L_N$ with $x \frown z$, a further collection of independent Poisson processes, denoted by $\{T_n^B(x, z) : n \geq 1\}$, each of them having rate B . We put an arrow from x to z at time $T_n^B(x, z)$ to indicate that a birth may occur. For each $x \in L_1$, we let $\{T_n^\delta(x) : n \geq 1\}$ be the arrival times of independent rate δ Poisson processes, and put a \times at site x at time $T_n^\delta(x)$ to indicate that a death may occur. Finally, to take into account the mass extinctions, we let $\{T_n^D(x) : n \geq 1\}$, $x \in L_N$, denote the arrival times of independent rate D Poisson processes, and put a \bullet at site x at time $T_n^D(x)$ to indicate that the square with center x gets empty.

Given an initial configuration η_0 , and the graphical representation introduced above, the process can be constructed as follows. If there is a particle

at site $x \in L_1$ at time $T_n^\beta(x, z)$ then site $z \in L_1$ becomes occupied if it is not already, that is the state of z flips from 0 to 1. In the same way, if there is a particle in $x \in L_N$ at time $T_n^B(x, z)$ then site $z \in L_N$ becomes occupied if it is not already. If there is a particle at site $x \in L_1$ at time $T_n^\delta(x)$, it is killed, that is the state of x flips from 1 to 0, regardless of the state of its neighbors. Finally, at time $T_n^D(x)$, $x \in L_N$, all the particles contained in the square with center x are killed. A nice feature of the graphical representation is that it allows us to couple processes with different parameters by using the same collections of Poisson processes. See [7, page 119] and [15, page 32].

With the graphical representation in hands, we can now prove the lemma. Let η_t^1 and η_t^2 denote the two-scale contact processes with parameters (B_1, β_1, N_1) and (B_2, β_2, N_2) , respectively, the death rates D and δ being the same for both processes, and assume that

$$B_1 \leq B_2, \quad \beta_1 \leq \beta_2 \quad \text{and} \quad N_1 \leq N_2.$$

Since in the case when $N_1 \neq N_2$ the processes η_t^1 and η_t^2 are defined on somewhat different spatial structures, the monotonicity with respect to N is not straightforward. To fix the problem, the basic idea is to inject the small squares (those of length side $2N_1 - 1$) into the large ones (those of length side $2N_2 - 1$) through a function ψ defined on \mathbb{Z}^d that maps L_{N_1} into L_{N_2} . We now consider the process η_t^2 constructed from the collections of Poisson processes introduced above for

$$B = B_2, \quad \beta = \beta_2 \quad \text{and} \quad N = N_2.$$

To prove that

$$P_0^1(|\eta_t^1| \geq 1 \text{ for all } t \geq 0) \leq P_0^2(|\eta_t^2| \geq 1 \text{ for all } t \geq 0)$$

where P_0^1 and P_0^2 denote the laws of the processes η_t^1 and η_t^2 starting with a single particle at site 0, the next step is to construct the process η_t^1 from the previous graphical representation in such a way that if $|\eta_t^1| \geq 1$ then $|\eta_t^2| \geq 1$ with probability 1.

For any $x \in \mathbb{Z}^d$, there exist a unique $q(x) \in \mathbb{Z}^d$ and a unique $r(x) \in (-N_1, N_1)^d$ such that

$$x = (2N_1 - 1)q(x) + r(x).$$

We then define a function $\psi : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ by setting

$$\psi(x) = \psi((2N_1 - 1)q(x) + r(x)) = (2N_2 - 1)q(x) + r(x).$$

See Figure 4 for a picture. The two-scale contact process η_t^1 can be constructed from the graphical representation of η_t^2 as follows. If there is a particle at site $x \in L_1$ at time $T_n^\beta(\psi(x), \psi(z))$, we toss a coin with success probability β_1/β_2 . If there is a success then site $z \in L_1$ becomes occupied if it is not already. In

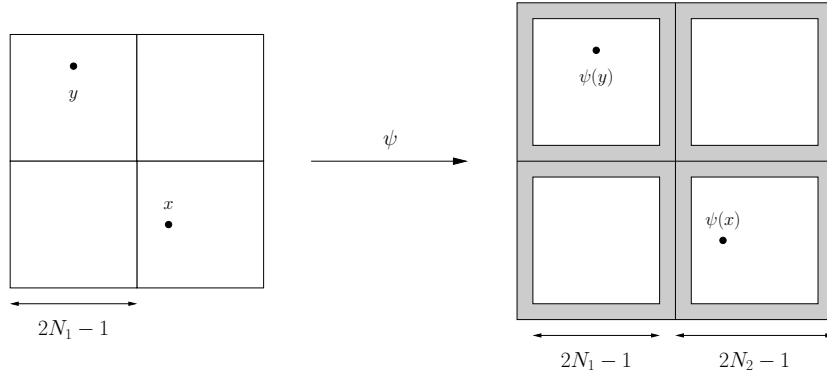


Figure 4. Picture of the coupling argument. The process η_t^1 (on the left-hand side) is constructed from the Poisson processes attached to the white squares of the right picture. The Poisson processes attached to the gray part of the right picture have no effect on the process η_t^1 .

the same way, if there is a particle at site $x \in L_N$ at time $T_n^B(\psi(x), \psi(z))$, we toss a coin with success probability B_1/B_2 . If there is a success then site $z \in L_N$ becomes occupied if it is not already. A particle at site $x \in L_1$ at time $T_n^\delta(\psi(x))$, if it exists, is killed regardless of the state of its neighbors. Finally, at time $T_n^D(\psi(x))$, $x \in L_N$, all the particles contained in the square with center x are killed.

When $N_1 = N_2$ (in particular $\psi = \text{id}$), one can check from the previous coupling that for any site $x \in \mathbb{Z}^d$, $\eta_t^1(x) \leq \eta_t^2(x)$ with probability 1 at any time $t \geq 0$, provided the inequalities hold at time 0. This proves the monotonicity with respect to B and β . When $N_1 < N_2$, the equalities

$$\psi((2N_1 - 1)z) = \psi((2N_2 - 1)z) \quad \text{for any } z \in \mathbb{Z}^d$$

imply that, if $\eta_0^1(x) \leq \eta_0^2(\psi(x))$ for any $x \in \mathbb{Z}^d$ at time 0, then $\eta_t^1(x) \leq \eta_t^2(\psi(x))$ for any $x \in \mathbb{Z}^d$ with probability 1 at any later time $t \geq 0$. This completes the proof of Lemma 1.1.

3. Proof of Theorems 1.1, 1.2 and 1.4

The proof essentially relies on coupling arguments, the aim is to compare η_t with well-known particle systems. The comparison is basically obtained thanks to a “destruction” of the spatial structure of each square. In other words, to figure out the evolution of the two-scale contact process, we introduce the

mesoscopic process $\bar{\eta}_t : \mathbb{Z}^d \rightarrow \{0, 1, \dots, (2N - 1)^d\}$ defined by

$$\bar{\eta}_t(x) = \sum_{z \in \mathcal{H}_x} \eta_t(z). \tag{3.1}$$

That is, $\bar{\eta}_t$ counts the number of particles in each square. Before going into the details of the proof, note that, in view of the loss of information due to the destruction of the spatial structure, the mesoscopic process $\bar{\eta}_t$ is not a Markov process.

Proof of Theorem 1.1

To deal with the convergence to the “all 0” configuration, it suffices to prove the result for the two-scale contact process with no mass extinction, that is when $D = 0$. The first step is to investigate the evolution of the mesoscopic process $\bar{\eta}_t$ introduced above. Let $\kappa_N = (2N - 1)^d$ denote the number of sites in each square. We observe that, when square \mathcal{H}_x has i particles, the number of empty sites with an occupied neighbor is bounded by $2di$. This implies that the value of $\bar{\eta}(x)$ flips from i to $i + 1$, $i = 0, 1, \dots, \kappa_N - 1$, at rate at most

$$2di\beta + B \sum_{x \sim z} \mathbb{1}\{\bar{\eta}(z) \neq 0\}. \tag{3.2}$$

Since the superposition of i independent rate 1 Poisson processes is a rate i Poisson process, the first death in a square with i particles occurs at rate i . This implies that, in the case when $D = 0$, the value of $\bar{\eta}(x)$ flips from i to $i - 1$, $i = 1, 2, \dots, \kappa_N$, at rate $i\delta$. In conclusion, the process $\bar{\eta}_t$ is stochastically smaller than the Markov process $\xi_t : \mathbb{Z}^d \rightarrow \{0, 1, \dots, \kappa_N\}$ whose pregenerator is defined, for any cylinder function f of the configuration ξ , by

$$\begin{aligned} \bar{\Omega}f(\xi) = & B \sum_{x \in \mathbb{Z}^d} \sum_{x \sim z} \mathbb{1}\{\xi(z) \neq 0\} \mathbb{1}\{\xi(x) \neq \kappa_N\} [f(\xi^{x,+}) - f(\xi)] \\ & + 2d\beta \sum_{x \in \mathbb{Z}^d} \xi(x) \mathbb{1}\{\xi(x) \neq \kappa_N\} [f(\xi^{x,+}) - f(\xi)] \\ & + \delta \sum_{x \in \mathbb{Z}^d} \xi(x) [f(\xi^{x,-}) - f(\xi)], \end{aligned}$$

where $\xi^{x,+}$ and $\xi^{x,-}$ are given by

$$\xi^{x,+}(z) = \begin{cases} \xi(z) + 1, & \text{if } z = x, \\ \xi(z) & \text{otherwise} \end{cases} \quad \text{and} \quad \xi^{x,-}(z) = \begin{cases} \xi(z) - 1, & \text{if } z = x, \\ \xi(z) & \text{otherwise.} \end{cases}$$

Theorem 1 in [2] implies that, when $B = 0$, the process ξ_t converges in distribution to the “all 0” configuration. In particular, for any $x \in \mathbb{Z}^d$ and any

$z \in \mathcal{H}_x$

$$\liminf_{t \rightarrow \infty} P^\eta(\eta_t(z) = 0) \geq \liminf_{t \rightarrow \infty} P^{\bar{\eta}}(\bar{\eta}_t(x) = 0) \geq \liminf_{t \rightarrow \infty} P^\xi(\xi_t(x) = 0) = 1$$

when $B = 0$. The proof relies on a comparison of the particle system viewed on suitable length and time scale with a 1-dependent oriented percolation process (see [5]), which provides a good enough rate of convergence of $P^\eta(\eta_t(z) = 0)$ so that a perturbation argument can be applied. Since the transition rates of η_t are continuous with respect to B , we can conclude that there is a small enough $B_1 > 0$ so that, when $B < B_1$, the two-scale contact process converges to the “all 0” configuration.

To prove the existence of a nontrivial stationary distribution when B is large, we start by noting that, when $\beta = 0$, the process $\bar{\eta}_t$ reduces to a contact process with parameter $B/(D + \delta)$. This, together with the monotonicity with respect to β , implies the existence of a $B_2 < \infty$ such that, when $B > B_2$, there is a nontrivial stationary distribution.

The existence of a $B_c \in [B_1, B_2]$ satisfying the statement of Theorem 1.1 then follows from the monotonicity with respect to B (see Lemma 1.1).

Proof of Theorem 1.2

Let $B \in (D\lambda_c, (D + \delta)\lambda_c)$. In view of the monotonicity with respect β , it suffices to prove that, when $\beta > 0$ is sufficiently small (respectively, when $\beta < \infty$ is sufficiently large), the process converges in distribution to the “all 0” configuration (respectively, there is a nontrivial stationary distribution). The first step is to analyse the behavior of the mesoscopic process in the two limiting cases $\beta = 0$ and $\beta = \infty$. When $\beta = 0$ (respectively, $\beta = \infty$ and $N \geq 2$), the process $\bar{\eta}_t$ reduces to a time change of the contact process with parameter

$$\frac{B}{D + \delta} < \lambda_c \quad \text{and} \quad \frac{B}{D} > \lambda_c$$

respectively. In particular, for any $x \in \mathbb{Z}^d$ and any $z \in \mathcal{H}_x$,

$$\liminf_{t \rightarrow \infty} P^\eta(\eta_t(z) = 0) \geq \liminf_{t \rightarrow \infty} P^{\bar{\eta}}(\bar{\eta}_t(x) = 0) = 1 \quad \text{when } \beta = 0,$$

and

$$\liminf_{t \rightarrow \infty} P^\eta(\eta_t(z) = 1) \geq \liminf_{t \rightarrow \infty} P^{\bar{\eta}}(\bar{\eta}_t(x) \neq 0) > 0 \quad \text{when } \beta = \infty.$$

The previous results are proved in [3,4] by comparing the particle system viewed on suitable length and time scale with oriented percolation. Since the transition rates of the two-scale contact process are continuous in β , the technique allows us to apply, as previously, a perturbation argument to obtain the existence of $\beta_1 > 0$ and $\beta_2 < \infty$ such that extinction occurs when $\beta < \beta_1$ while survival occurs when $\beta > \beta_2$.

Proof of Theorem 1.4

Let $\delta = 0$. To begin with, we observe that, since $\delta = 0$, a square is occupied by a particle if and only if its center is occupied, i.e.,

$$\bar{\eta}_t(x) \neq 0 \quad \text{if and only if} \quad \eta_t((2N - 1)x) = 1.$$

In particular, the existence of a nontrivial stationary distribution does not depend on the value of the microscopic birth rate β . In other respects, when $\beta = 0$, the process $\bar{\eta}_t$ reduces to a time change of the contact process with parameter B_N/D_N . This implies Theorem 1.4.

4. Proof of Theorem 1.3

The aim of this section is to prove that, when $2d(B + \beta) < D + \delta$, the process converges to the “all 0” configuration. As previously, it suffices to establish the result for a process ξ_t which is stochastically larger than the mesoscopic process introduced in (3.1). The convergence to the “all 0” configuration will be deduced from an ergodicity criterion established in [6].

The process $\xi_t : \mathbb{Z}^d \rightarrow \{0, 1, \dots, \kappa_N\}$, $\kappa_N = (2N - 1)^d$, we will consider is the Markov process whose pregenerator $\bar{\Omega}$ is defined, for any cylinder function f of the configuration ξ , by

$$\begin{aligned} \bar{\Omega}f(\xi) &= 2dB \sum_{x,y \in \mathbb{Z}^d} p(x,y)\xi(y)\mathbb{1}\{\xi(x) \neq \kappa_N\}[f(\xi^{x,+}) - f(\xi)] \\ &\quad + 2d\beta \sum_{x \in \mathbb{Z}^d} \xi(x)\mathbb{1}\{\xi(x) \neq \kappa_N\}[f(\xi^{x,+}) - f(\xi)] \\ &\quad + \delta \sum_{x \in \mathbb{Z}^d} \xi(x)[f(\xi^{x,-}) - f(\xi)] + D \sum_{x \in \mathbb{Z}^d} [f(\xi^x) - f(\xi)] \end{aligned}$$

where $p(x, y) = 1/2d$ if $x \sim y$, and $= 0$ otherwise, and where ξ^x is the configuration obtained from the configuration ξ by killing all the particles at site x . In words, each particle on the lattice gives birth to a particle that is sent to one of the nearest neighbors (respectively, within the same site) at rate $2dB$ (respectively, $2d\beta$). In any case, when a particle is sent to a site already occupied by κ_N particles, the birth is suppressed. Moreover, all the particles belonging to the same site die individually at rate δ or simultaneous at rate D . The argument given in the previous section (see (3.2) above) implies that the process ξ_t is stochastically larger than the mesoscopic process introduced in (3.1). In particular, Theorem 1.3 will follow by proving that the process ξ_t converges to the “all 0” configuration.

The investigation of the stationary distributions of the auxiliary process ξ_t can be done through techniques introduced in [6]. We also refer to [1] for a

similar approach. First of all, given a constant $M > 1$, we set

$$k_x = \sum_{n=0}^{\infty} M^{-n} p^{(n)}(x, 0) \quad \text{for all } x \in \mathbb{Z}^d. \tag{4.1}$$

Since $p(x, y)$ is translation invariant with $p(x, x) = 0$, we have

$$\sum_{y \in \mathbb{Z}^d} p(x, y) k_y \leq M k_x \quad \text{and} \quad \sum_{x \in \mathbb{Z}^d} k_x < +\infty. \tag{4.2}$$

See Remark 14.9 in [6] for a rigorous proof. Now, given a site $x \in \mathbb{Z}^d$ and two configurations ξ_1 and ξ_2 , we set

$$q_x(\xi_1) = \xi_1(x) k_x \quad \text{and} \quad q_x(\xi_1, \xi_2) = |\xi_1(x) - \xi_2(x)| k_x$$

where k_x is defined by equation (4.1). For any integer $n \geq 1$, we set $\Lambda_n = \{-n, \dots, n\}^d$ and let ξ_t^n denote the process ξ_t conditioned so that $\xi_t^n \equiv 0$ outside Λ_n . Finally, we let $\bar{\Omega}_n$ denote the pregenerator of ξ_t^n . The following theorem is a straightforward consequence of Theorems 13.8 and 14.3 in [6] for a Markov process with compact state space.

Theorem 4.1. *Assume that for all $1 \leq n \leq m$, there is a coupling of $\bar{\Omega}_n$ and $\bar{\Omega}_m$, denoted by $\bar{\Omega}_{n,m}$, such that for any $z \in \Lambda_n$ and any configurations ξ_1 and ξ_2*

$$\bar{\Omega}_{n,m} q_z(\xi_1, \xi_2) \leq \sum_{y \in \Lambda_n} c_{yz} q_y(\xi_1, \xi_2) + \sum_{y \in \Lambda_m \setminus \Lambda_n} g_{yz} q_y(\xi_2) \tag{4.3}$$

with $c_{yz} \geq 0$ and $g_{yz} \geq 0$ for all $y, z \in \mathbb{Z}^d$, $y \neq z$, and such that

$$\lim_{m \geq n \rightarrow \infty} \sup_{y \in \Lambda_m} \sum_{z \in \Lambda_n} (c_{yz} + g_{yz}) < +\infty, \tag{4.4}$$

$$\lim_{m \geq n \rightarrow \infty} \sup_{y \in \Lambda_m} \sum_{z \in \Lambda_n} c_{yz} \leq -\alpha \quad \text{for some } \alpha > 0, \tag{4.5}$$

$$\lim_{m \geq n \rightarrow \infty} \sup_{y \in \Lambda_m} \sum_{z \in \Lambda_n} |c_{yz}| \leq K \quad \text{for some } K > 0. \tag{4.6}$$

Then the process ξ_t has at most one stationary distribution.

Since the “all 0” configuration is an absorbing state for the Markov process ξ_t it suffices to find a coupling $\bar{\Omega}_{n,m}$ of $\bar{\Omega}_n$ and $\bar{\Omega}_m$ which satisfies (4.4)–(4.6) to conclude the proof of Theorem 1.3. We will prove the result for the basic coupling. To lighten our calculations, we set

$$a_i(x, y) = \xi_i(y) \mathbb{1}\{\xi_i(x) \neq \kappa_N\} \quad \text{for } i = 1, 2 \text{ and } x, y \in \mathbb{Z}^d.$$

We define the coupling $\bar{\Omega}_{n,m}$ as the sum of $\bar{\Omega}_{n,m}^i$, $i = 1, 2, 3, 4$, with

$$\begin{aligned} \bar{\Omega}_{n,m}^1 f(\xi_1, \xi_2) &= 2dB \sum_{x,y \in \Lambda_n} p(x,y)(a_1(x,y) \wedge a_2(x,y))[f(\xi_1^{x,+}, \xi_2^{x,+}) - f(\xi_1, \xi_2)] \\ &\quad + 2dB \sum_{x,y \in \Lambda_n} p(x,y)(a_1(x,y) - a_2(x,y))^+ [f(\xi_1^{x,+}, \xi_2) - f(\xi_1, \xi_2)] \\ &\quad + 2dB \sum_{x,y \in \Lambda_n} p(x,y)(a_2(x,y) - a_1(x,y))^+ [f(\xi_1, \xi_2^{x,+}) - f(\xi_1, \xi_2)] \\ &\quad + 2dB \sum_{x \in \Lambda_m \setminus \Lambda_n} \sum_{y \in \Lambda_m} p(x,y)a_2(x,y)[f(\xi_1, \xi_2^{x,+}) - f(\xi_1, \xi_2)] \\ &\quad + 2dB \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_m \setminus \Lambda_n} p(x,y)a_2(x,y)[f(\xi_1, \xi_2^{x,+}) - f(\xi_1, \xi_2)] \end{aligned}$$

describing the births originated from neighboring sites, with

$$\begin{aligned} \bar{\Omega}_{n,m}^2 f(\xi_1, \xi_2) &= 2d\beta \sum_{x \in \Lambda_n} (a_1(x,x) \wedge a_2(x,x))[f(\xi_1^{x,+}, \xi_2^{x,+}) - f(\xi_1, \xi_2)] \\ &\quad + 2d\beta \sum_{x \in \Lambda_n} (a_1(x,x) - a_2(x,x))^+ [f(\xi_1^{x,+}, \xi_2) - f(\xi_1, \xi_2)] \\ &\quad + 2d\beta \sum_{x \in \Lambda_n} (a_2(x,x) - a_1(x,x))^+ [f(\xi_1, \xi_2^{x,+}) - f(\xi_1, \xi_2)] \\ &\quad + 2d\beta \sum_{x \in \Lambda_m \setminus \Lambda_n} a_2(x,x)[f(\xi_1, \xi_2^{x,+}) - f(\xi_1, \xi_2)] \end{aligned}$$

describing the births within a given site, with

$$\begin{aligned} \bar{\Omega}_{n,m}^3 f(\xi_1, \xi_2) &= \delta \sum_{x \in \Lambda_n} (\xi_1(x) \wedge \xi_2(x))[f(\xi_1^{x,-}, \xi_2^{x,-}) - f(\xi_1, \xi_2)] \\ &\quad + \delta \sum_{x \in \Lambda_n} (\xi_1(x) - \xi_2(x))^+ [f(\xi_1^{x,-}, \xi_2) - f(\xi_1, \xi_2)] \\ &\quad + \delta \sum_{x \in \Lambda_n} (\xi_2(x) - \xi_1(x))^+ [f(\xi_1, \xi_2^{x,-}) - f(\xi_1, \xi_2)] \\ &\quad + \delta \sum_{x \in \Lambda_m \setminus \Lambda_n} \xi_2(x)[f(\xi_1, \xi_2^{x,-}) - f(\xi_1, \xi_2)] \end{aligned}$$

describing the individual deaths, and with

$$\bar{\Omega}_{n,m}^4 f(\xi_1, \xi_2) = D \sum_{x \in \Lambda_m} [f(\xi_1^x, \xi_2^x) - f(\xi_1, \xi_2)]$$

describing mass extinctions in a given site. For sites $x, y, z \in \mathbb{Z}^d$, we set

$$b_z(x, y) = (a_1(x, y) - a_2(x, y))^+ [q_z(\xi_1^{x,+}, \xi_2) - q_z(\xi_1, \xi_2)] \\ + (a_2(x, y) - a_1(x, y))^+ [q_z(\xi_1, \xi_2^{x,+}) - q_z(\xi_1, \xi_2)].$$

First of all, we observe that

$$q_z(\xi_1^{x,+}, \xi_2) = q_z(\xi_1, \xi_2^{x,-}) = \begin{cases} q_z(\xi_1, \xi_2) + k_z & \text{when } \xi_1(x) \geq \xi_2(x) \text{ and } x = z, \\ q_z(\xi_1, \xi_2) - k_z & \text{when } \xi_1(x) < \xi_2(x) \text{ and } x = z, \end{cases} \quad (4.7)$$

and is equal to zero when $x \neq z$, while

$$q_z(\xi_1, \xi_2^{x,+}) = q_z(\xi_1^{x,-}, \xi_2) = \begin{cases} q_z(\xi_1, \xi_2) + k_z & \text{when } \xi_1(x) \leq \xi_2(x) \text{ and } x = z, \\ q_z(\xi_1, \xi_2) - k_z & \text{when } \xi_1(x) > \xi_2(x) \text{ and } x = z \end{cases} \quad (4.8)$$

and is equal to zero when $x \neq z$. In particular, by assuming that $\xi_1(x) > \xi_2(x)$ and by decomposing according to whether $\xi_1(x)$ is different from or equal to κ_N , we obtain

$$b_z(x, y) = (\xi_1(y) - \xi_2(y))k_z \mathbb{1}\{\xi_1(x) \neq \kappa_N\} - \xi_2(y)k_z \mathbb{1}\{\xi_1(x) = \kappa_N\} \\ \leq |\xi_1(y) - \xi_2(y)|k_z = q_y(\xi_1, \xi_2)k_z/k_y$$

when $x = z$, and $b_z(x, y) = 0$ when $x \neq z$. By symmetry, the previous inequality holds as well in the case $\xi_1(x) < \xi_2(x)$. Since it is trivial in the case $\xi_1(x) = \xi_2(x)$, we conclude that

$$b_z(x, y) \leq q_y(\xi_1, \xi_2)k_z/k_y \quad \text{if } x = z \quad \text{and} \quad b_z(x, y) = 0 \quad \text{if } x \neq z. \quad (4.9)$$

By using (4.9), we obtain, for any site $z \in \Lambda_n \subset \Lambda_m$,

$$\bar{\Omega}_{n,m}^1 q_z(\xi_1, \xi_2) = 2dB \sum_{y \in \Lambda_n} p(z, y) b_z(z, y) \\ + 2dB \sum_{y \in \Lambda_m \setminus \Lambda_n} p(z, y) a_2(z, y) [q_z(\xi_1, \xi_2^{z,+}) - q_z(\xi_1, \xi_2)] \\ \leq 2dB \sum_{y \in \Lambda_n} p(z, y) q_y(\xi_1, \xi_2) k_z/k_y \\ + 2dB \sum_{y \in \Lambda_m \setminus \Lambda_n} p(z, y) q_y(\xi_2) k_z/k_y.$$

Now, assume that $\xi_1(z) > \xi_2(z)$ for some $z \in \Lambda_n$. From (4.7) and (4.8), it follows that

$$(\bar{\Omega}_{n,m}^2 + \bar{\Omega}_{n,m}^3) q_z(\xi_1, \xi_2) = 2d\beta [(a_1(z, z) - a_2(z, z))^+ - (a_2(z, z) - a_1(z, z))^+] k_z \\ - \delta [(\xi_1(z) - \xi_2(z))^+ - (\xi_2(z) - \xi_1(z))^+] k_z \\ = (2d\beta - \delta) q_z(\xi_1, \xi_2)$$

when $\xi_1(z) \neq \kappa_N$, and

$$(\bar{\Omega}_{n,m}^2 + \bar{\Omega}_{n,m}^3)q_z(\xi_1, \xi_2) = -2d\beta q_z(\xi_2) - \delta q_z(\xi_1, \xi_2) \leq (2d\beta - \delta)q_z(\xi_1, \xi_2)$$

when $\xi_1(z) = \kappa_N$. The same holds when $\xi_1(z) < \xi_2(z)$. In particular,

$$(\bar{\Omega}_{n,m}^2 + \bar{\Omega}_{n,m}^3)q_z(\xi_1, \xi_2) \leq (2d\beta - \delta)q_z(\xi_1, \xi_2)$$

in any case since both members of the inequality are equal to 0 when $\xi_1(z) = \xi_2(z)$. Finally, by observing that $q_z(\xi_1^x, \xi_2^x) = q_z(\xi_1, \xi_2)$ when $x \neq z$, and is equal to zero when $x = z$, we have

$$\bar{\Omega}_{n,m}^4 q_z(\xi_1, \xi_2) = -Dq_z(\xi_1, \xi_2).$$

Putting things together, we get the upper bound

$$\bar{\Omega}_{n,m} q_z(\xi_1, \xi_2) \leq \sum_{y \in \Lambda_n} c_{yz} q_y(\xi_1, \xi_2) + \sum_{y \in \Lambda_m \setminus \Lambda_n} g_{yz} q_y(\xi_2)$$

where the coefficients c_{yz} and g_{yz} are given by

$$c_{yz} = \begin{cases} 2d\beta - D - \delta, & \text{if } y = z, \\ 2dBp(z, y)k_z/k_y, & \text{if } y \neq z \end{cases} \quad \text{and} \quad g_{yz} = 2dBp(z, y)k_z/k_y.$$

By (4.1) and (4.2), for any site $y \in \Lambda_m$ and any constant $M > 1$,

$$\sum_{z \in \Lambda_n} c_{yz} = 2dB \sum_{z \in \Lambda_n} p(z, y)k_z/k_y + 2d\beta - D - \delta \leq 2d(\beta + MB) - (D + \delta).$$

In particular, condition (4.5) in Theorem 4.1 holds whenever $2d(B + \beta) < D + \delta$. In other respects, conditions (4.4) and (4.6) are trivial. This completes the proof of Theorem 1.3.

5. Proof of Theorem 1.5

In this section, we show that increasing N promotes survival of the process without mass extinction, that is when $\delta = 1$ and $D_N = 0$. The strategy is to compare the particle system viewed on suitable length and time scales with a 1-dependent oriented percolation process on

$$\mathcal{L} = \{(z, n) \in \mathbb{Z}^2 : z + n \text{ is even and } n \geq 0\}$$

with parameter $1 - \varepsilon$. Each site $(z, n) \in \mathcal{L}$ is associated with a random variable $\omega(z, n) \in \{0, 1\}$ which indicates whether the site is open (1) or closed (0), and satisfies

$$P(\omega(z_i, n_i) = 1 \text{ for } 1 \leq i \leq m) = (1 - \varepsilon)^m$$

whenever $\|(z_i, n_i) - (z_j, n_j)\|_\infty > 1$ for $i \neq j$. A site (z, n) is said to be wet (at level n) if there is a sequence $z_0, z_1, \dots, z_n = z$ such that

1. For $i = 0, 1, \dots, n - 1$, we have $|z_{i+1} - z_i| = 1$.
2. For $i = 0, 1, \dots, n$, site (z_i, i) is open, that is $\omega(z_i, i) = 1$.

To compare the two-scale contact process with oriented percolation, we let T_N and γ be two constants to be fixed later, and say that site $(z, n) \in \mathcal{L}$ is occupied if $(2N - 1)ze_1$ is occupied by a particle at least γT_N units of time between time nT_N and time $(n + 1)T_N$, i.e.,

$$f_N(z, n) = \frac{1}{T_N} \int_{nT_N}^{(n+1)T_N} \mathbb{1}\{\eta_t((2N - 1)ze_1) = 1\} dt \geq \gamma.$$

Theorem 1.5, whose details of the proof are supplied after the proof of Lemma 5.5, is a consequence of the following lemma.

Lemma 5.1. *Assume that $B_N > \exp(-bN^d)$ and $\beta > \lambda_c$. If $b > 0$ is sufficiently small then, for any $\varepsilon > 0$, there exists a large enough N such that the set of occupied sites dominates the set of wet sites in a 1-dependent oriented percolation process with parameter $1 - \varepsilon$.*

The proof of Lemma 5.1 essentially relies on estimates on the extinction time of the process restricted to a single square (when the mesoscopic birth rate B_N is equal to 0).

The contact process on a finite set

To describe the behavior of the process in an isolated square, we consider the contact process ζ_t^N evolving on $\mathcal{B}_N = (-N, N)^d$ and let

$$\tau_N = \inf \{t \geq 0 : \zeta_t^N \equiv 0\}$$

denote the extinction time of the process. Since ζ_t^N is a finite state space Markov process, it converges in distribution to its single absorbing state, the “all 0” configuration, so that τ_N is finite with probability 1. The aim is to prove that, when N is large, the process starting with a single particle at site 0 dies out exponentially fast or lives an exponentially long time. Moreover, in the latter case, site 0 is occupied by a particle a positive fraction of time. To avoid cumbersome notations, we prove the result in two dimensions only but the proof is easy to extend to any dimension $d \geq 1$. First of all, we tile \mathcal{B}_N with $a \times a$ squares by setting

$$\pi(w) = (aw_1, aw_2), \quad E = (-a/2, a/2]^2 \quad \text{and} \quad E(w) = \pi(w) + E$$

for $w \in \mathbb{Z}^2$, and assume, for more convenience, that $q_N = (2N - 1)/a$ is an integer. We say that a site w with $E(w) \subset \mathcal{B}_N$ is good if the square $E(w)$

contains at least \sqrt{a} particles, and that a configuration $\zeta^N : \mathcal{B}_N \rightarrow \{0, 1\}$ is good if, in the configuration ζ^N , there are at least \sqrt{N} good sites w . Let \mathcal{G}_N denote the set of good configurations, and Ω_N the event

$$\Omega_N = \{\zeta_t^N \in \mathcal{G}_N \text{ for some } t \geq 0\}.$$

Let P_0^N denote the law of the contact process ζ_t^N starting with a single particle at site 0. The first step is to prove that on the event Ω_N^c , the complement of Ω_N , the process dies out quickly.

Lemma 5.2. *Assume that $\beta > \lambda_c$. Then, for a sufficiently large,*

$$P_0^N(\Omega_N^c; \tau_N > t) \leq \varepsilon_N + C_1 \exp(-\gamma_1 t)$$

with $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$ and for suitable $C_1 < \infty$ and $\gamma_1 > 0$.

Proof. The proof is divided into two steps. We will decompose the event to be estimated according to whether the stopping time π_N is (i) finite or (ii) infinite, where

$$\pi_N = \inf \{t \geq 0 : \zeta_t^N(x) = 1 \text{ for some } x \in \mathcal{B}_N \setminus \mathcal{B}_{N-1}\}$$

denotes the first time the process reaches the boundary of \mathcal{B}_N . In either case, we will use the following property: Let ζ_t denote the contact process with parameter β on \mathbb{Z}^d starting with a single particle at site 0. Then, since none of the particles (for the process ζ_t^N) is sent outside \mathcal{B}_N by time π_N , the processes ζ_t and ζ_t^N can be constructed on the same probability space (using the same graphical representation) in such a way that $\zeta_t = \zeta_t^N$ a.s. at any time $t \leq \pi_N$.

(i) $\pi_N < \infty$. In this case, we will prove that

$$P_0^N(\Omega_N^c; \pi_N < \infty) \leq \varepsilon_N \tag{5.1}$$

with $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$. First, since the contact process ζ_t^N is supercritical ($\beta > \lambda_c$), the probability that $\pi_N < \infty$ is larger (uniformly in N) than some positive constant, which allows us to condition on the event that $\pi_N < \infty$ regardless of the value of N . By using that $\zeta_t^N = \zeta_t$ a.s. for all $t \leq \pi_N$ and by applying the Shape Theorem in [15, p. 128] to the unrestricted contact process ζ_t , we obtain the existence of a constant $c > 0$ such that for any $\varepsilon > 0$

$$\lim_{N \rightarrow \infty} P(|N^{-1}\pi_N - c| > \varepsilon \mid \pi_N < \infty) = 0. \tag{5.2}$$

Roughly speaking, (5.2) indicates that π_N is either of the order of cN or infinite. Now, let θ_t denote the contact process on \mathbb{Z}^d starting from the “all occupied” configuration and constructed from the same graphical representation as ζ_t .

Then, (5.2) and another application of the Shape Theorem imply that there exists $C > 0$ such that

$$\begin{aligned} & \lim_{N \rightarrow \infty} P(\zeta_{\pi_N}^N(x) \neq \theta_{\pi_N}(x) \text{ for some } x \in (-CN, CN)^2 \mid \pi_N < \infty) \\ &= \lim_{N \rightarrow \infty} P(\zeta_{\pi_N}(x) \neq \theta_{\pi_N}(x) \text{ for some } x \in (-CN, CN)^2 \mid \pi_N < \infty) = 0. \end{aligned} \tag{5.3}$$

Moreover, by attractivity, the distribution of θ_{π_N} dominates the upper invariant measure $\bar{\mu}$ of the basic contact process on \mathbb{Z}^d . Now, let $w \in \mathbb{Z}^2$ with $E(w) \subset (-CN, CN)^2$ and choose a set of a points $x_1, x_2, \dots, x_a \in E(w)$ such that

$$\|x_i - x_j\|_2 \geq \log a \quad \text{for all } i, j \in \{1, 2, \dots, a\}, i \neq j,$$

where $\|x_i - x_j\|_2$ denotes the Euclidean distance between x_i and x_j . Then, by applying Theorem 4.20, [14, p. 41] (see also Theorem 1.7 in [9]) to the invariant measure $\bar{\mu}$ with $f(\zeta) = \zeta(x_i)$ and $g(\zeta) = \zeta(x_j)$, $i, j \in \{1, 2, \dots, a\}, i \neq j$, we get

$$\begin{aligned} & |\bar{\mu}(\zeta : \zeta(x_i) = \zeta(x_j) = 1) - \bar{\mu}(\zeta : \zeta(x_i) = 1)\bar{\mu}(\zeta : \zeta(x_j) = 1)| \\ & \leq C_2 \exp(-\gamma_2 \|x_i - x_j\|_2) \leq C_2 a^{-\gamma_2} \end{aligned} \tag{5.4}$$

for some $C_2 < \infty$ and $\gamma_2 > 0$. Since $\bar{\mu}(\zeta : \zeta(x) = 1) \geq \rho > 0$, we can deduce from (5.4) that

$$\lim_{a \rightarrow \infty} \bar{\mu}(w \text{ is good}) = \lim_{a \rightarrow \infty} \bar{\mu}(E(w) \text{ contains at least } \sqrt{a} \text{ particles}) = 1$$

while holding the ratio N/a constant. In particular, the number of $E(w) \subset (-CN, CN)^2$ containing at least \sqrt{a} particles for a large is larger than \sqrt{N} with probability close to 1 when the parameter N is sufficiently large. Putting things together, we obtain that for a large and conditioned on the event that $\pi_N < \infty$, the event Ω_N occurs with probability close to 1 when N is large:

$$\lim_{N \rightarrow \infty} P_0^N(\Omega_N \mid \pi_N < \infty) = 1.$$

This completes the proof of (5.1).

(ii) $\pi_N = \infty$. In this case, we will prove that

$$P_0^N(\tau_N > t; \pi_N = \infty) \leq C_1 \exp(-\gamma_1 t) \tag{5.5}$$

for suitable $C_1 < \infty$ and $\gamma_1 > 0$. Let τ be the extinction time of ζ_t . Then, Theorem 2.30 in [15] implies that there exist constants $C_1 < \infty$ and $\gamma_1 > 0$ such that

$$P_0(t < \tau < \infty) \leq C_1 \exp(-\gamma_1 t) \tag{5.6}$$

where P_0 denotes the law of the process ζ_t (starting with a single particle at site 0). Since on the event $\{\pi_N = \infty\}$ we have $\zeta_t = \zeta_t^N$ a.s. at any time $t \geq 0$, we obtain the inclusions

$$\{\pi_N = \infty\} \subset \{\tau = \tau_N\} \subset \{\tau < \infty\}, \tag{5.7}$$

so that $P_0^N(\tau_N > t; \pi_N = \infty) \leq P_0(\tau > t; \tau < \infty)$. This together with (5.6) implies (5.5).

Finally, by decomposing the event of interest according to whether π_N is finite or infinite, it follows from (5.1) and (5.5) that

$$\begin{aligned} P_0^N(\Omega_N^c; \tau_N > t) &\leq P_0^N(\Omega_N^c; \pi_N < \infty) + P_0^N(\tau_N > t; \pi_N = \infty) \\ &\leq \varepsilon_N + C_1 \exp(-\gamma_1 t). \end{aligned}$$

This completes the proof. □

The next step is to prove that, on the event Ω_N , the process ζ_t^N lives an exponentially long time, and to estimate the amount of time site 0 is occupied by a particle.

Lemma 5.3. *Let $\beta > \lambda_c$ and $T_N = \exp(cN^2)$. Then there exist $c > 0$ and $a > 0$ such that*

$$P_0^N(\tau_N < 3T_N; \Omega_N) \leq C_3 \exp(-\gamma_3 q_N^2) \tag{5.8}$$

for suitable constants $C_3 < \infty$ and $\gamma_3 > 0$. Moreover, on the event that $\tau_N \geq 3T_N$,

$$\lim_{N \rightarrow \infty} P_0^N \left(\frac{1}{T_N} \int_s^{s+T_N} \mathbb{1}\{\zeta_t^N(0) = 1\} dt \geq \gamma \right) = 1 \tag{5.9}$$

for some $\gamma > 0$ and all $0 < s < T_N$.

Proof. The first step is to set the parameter a sufficiently large so that the process viewed on the macroscopic $q_N \times q_N$ lattice dominates a (supercritical) oriented percolation process. Let $\varepsilon > 0$ small. Since $\beta > \lambda_c$ there exist a $\Gamma > 0$ and a large $a > 0$, fixed from now on, such that the following holds: If $w \sim w'$ with $E(w), E(w') \subset \mathcal{B}_N$, and w is good at time 0, then

$$P(w' \text{ is good at time } \Gamma a) \geq 1 - \varepsilon. \tag{5.10}$$

See [3]. Then, we use an idea in [15, p. 76], to compare the contact process on \mathcal{B}_N with an oriented percolation process with parameter $1 - \varepsilon$ evolving in a linear tube of length q_N^2 embedded in \mathcal{B}_N . The basic construction is shown in Figure 5, left picture. The comparison is obtained by assuming that births through the continuous thick lines of the picture are not allowed, i.e., if $x, z \in \mathcal{B}_N$ with $x \sim z$ belong to adjacent $a \times a$ squares whose joint side is drawn in thick line,

then we suppress the interactions between x and z . Let W_n denote the set of wet sites in the percolation process at level n and

$$X_n = \{w \in \mathbb{Z}^2 : E(w) \subset \mathcal{B}_N \text{ and } w \text{ is good at time } n\Gamma a\}.$$

Then, the lower bound in (5.10) implies that the set of open sites in the percolation process can be chosen in such a way that $W_n \subset X_n$ at any level $n \geq 1$ provided $W_0 \subset X_0$ (see [5]). In particular, the stopping time τ_N is bounded from below by the extinction time of the percolation process. Since on the event Ω_N there is an integer $k \geq 0$ such that the set of open sites at level k can be made arbitrarily large by choosing N large enough (recall that $\zeta_{k\Gamma a} \in \mathcal{G}_N$ implies that $|W_k| \geq \sqrt{N}$), inequality (5.8) follows from the analogous result for the percolation process evolving on the segment $\{0, 1, \dots, q_N^2 - 1\}$. A proof can be found in [15, pp. 77–78] (see also Theorem 2 in [8]). Since in addition the density of wet sites is positive, denoting by k_N the integer part of $T_N/\Gamma a$ we obtain

$$\liminf_{N \rightarrow \infty} \frac{1}{k_N} \sum_{k=j}^{j+k_N-1} \mathbb{1}\{0 \in W_k\} = \ell > 0 \quad \text{a.s.}$$

for all integer $j \leq k_N$. In other respects, if $0 \in W_k$ then, with some positive probability that depends on a but not on the parameter N , site 0 is occupied by a particle at least one unit of time between time $k\Gamma a$ and time $(k + 1)\Gamma a$. This implies that

$$\lim_{N \rightarrow \infty} P_0^N \left(\frac{1}{T_N} \int_s^{s+T_N} \mathbb{1}\{\zeta_t^N(0) = 1\} dt \geq \gamma \right) = 1$$

for some $\gamma > 0$ and all $0 < s < T_N$. This completes the proof. □

Lemma 5.4. *Let $\beta > \lambda_c$ and $T_N = \exp(cN^2)$ as in Lemma 5.3. Then,*

$$P_0^N(t < \tau_N < 3T_N) \leq \varepsilon'_N + C_1 \exp(-\gamma_1 t)$$

with $\varepsilon'_N \rightarrow 0$ as $N \rightarrow \infty$.

Proof. By decomposing the event to be estimated according to whether Ω_N occurs or not, it follows from Lemma 5.2 and Lemma 5.3 that

$$\begin{aligned} P_0^N(t < \tau_N < 3T_N) &\leq P_0^N(\tau_N > t; \Omega_N^c) + P_0^N(\tau_N < 3T_N; \Omega_N) \\ &\leq \varepsilon_N + C_1 \exp(-\gamma_1 t) + C_3 \exp(-\gamma_3 q_N^2) \\ &= \varepsilon'_N + C_1 \exp(-\gamma_1 t) \end{aligned}$$

with $\varepsilon'_N \rightarrow 0$ as $N \rightarrow \infty$. This completes the proof of the lemma. □

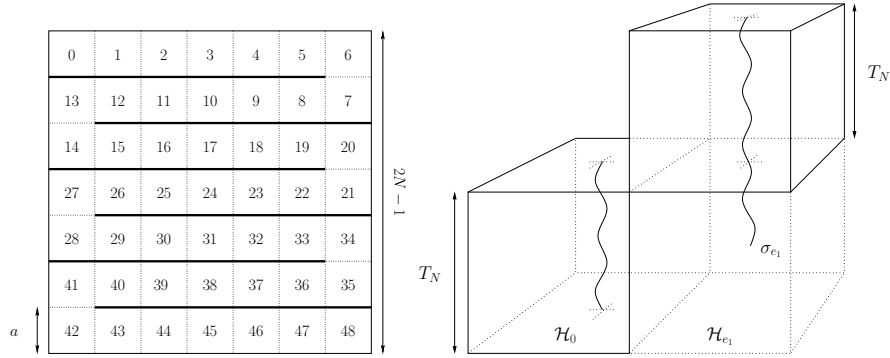


Figure 5. Pictures of the rescaling arguments.

Percolation structure. Proof of Theorem 1.5

With Lemmas 5.2-5.4 in hands, we are now ready to prove Lemma 5.1 and Theorem 1.5. First of all, since the evolution rules of η_t are invariant by translation of vector $x \in L_N$, it suffices to prove the following statement: If site $(0, 0)$ is occupied then, with probability at least $1 - \varepsilon$ when N is sufficiently large, site $(1, 1)$ is occupied. See Figure 5 for a picture. To facilitate the application of the restart argument described in Lemma 5.5 below, we prove the result for the process η_t modified so that births at the mesoscopic level are not allowed in nonempty squares, that is squares in which there is at least one particle. Each time a particle at site 0 gives birth to a particle which is sent to site $(2N - 1)e_1$, we call this particle a good particle if its offspring lives at least $3T_N$ units of time, and let

$$\sigma_{e_1} = \inf \{t \geq 0 : \bar{\eta}_s(e_1) \neq 0 \text{ for any } t \leq s \leq t + 3T_N\}$$

denote the first time a good particle is sent to site $(2N - 1)e_1$. In the definition of σ_{e_1} the process $\bar{\eta}_t$ is the mesoscopic process introduced in Section 3.

Lemma 5.5. *Let $T_N = \exp(cN^2)$ and $\gamma > 0$ as in Lemma 5.3. If $B_N > \exp(-bN^2)$ for some $0 < b < c$ and site $(0, 0)$ is occupied then*

$$P^n((1, 1) \text{ is occupied}) \geq 1 - \varepsilon \quad \text{for all } N \text{ sufficiently large.}$$

Proof. The first step is to prove that, with probability close to 1 for N large, a good particle is sent to site $(2N - 1)e_1$ by time T_N . For any $i \geq 1$, we introduce the stopping times

$$\rho_i = \inf \{t \geq \bar{\rho}_{i-1} : \bar{\eta}_t(e_1) = 1\} \quad \text{and} \quad \bar{\rho}_i = \inf \{t \geq \rho_i : \bar{\eta}_t(e_1) = 0\}$$

with the convention $\bar{\rho}_0 = 0$. That is, ρ_i is the i th time a particle born at site 0 is sent to the center of square \mathcal{H}_{e_1} and $\bar{\rho}_i$ the i th time square \mathcal{H}_{e_1} becomes empty. Denoting by

$$K = \inf \{i \geq 1 : \bar{\rho}_i - \rho_i > 3T_N\}$$

the number of trials needed to give birth to a good particle, we get $\rho_K = \sigma_{e_1}$. Note that, since we deal with the process modified so that births at the mesoscopic level are not allowed in nonempty squares, we have for all $n \geq 2$

$$P^\eta(K = 1) = P^\eta(K = n \mid K \geq n) \tag{5.11}$$

which is also the probability that the contact process restricted to \mathcal{H}_{e_1} starting with a single particle at site $(2N - 1)e_1$ lives (at least) until time $3T_N$. This implies that K is geometrically distributed. Now, by decomposing according to the value of K , we get

$$P^\eta(\sigma_{e_1} > T_N) = \sum_{n=1}^{\infty} P^\eta(\rho_n > T_N; K = n). \tag{5.12}$$

Let $\bar{r}_i = \bar{\rho}_i - \rho_i$ be the time it takes for square \mathcal{H}_{e_1} to get empty after the i th trial, and r_i be the amount of time site 0 is occupied by a particle between time $\bar{\rho}_{i-1}$ and time ρ_i . Since site 0 is occupied at least γT_N units of time between 0 and T_N , on the event $\{\rho_n > T_N\}$, we have

$$\sum_{i=1}^{n-1} (\bar{r}_i + r_i) \geq \gamma T_N,$$

which implies that \bar{r}_i or r_i is larger than $\gamma T_N/2n$ for some $i \in \{1, 2, \dots, n - 1\}$. Reporting in equation (5.12), we obtain

$$\begin{aligned} P^\eta(\sigma_{e_1} > T_N) &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} P^\eta\left(r_i > \frac{\gamma T_N}{2n}; K = n\right) \\ &\quad + \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} P^\eta\left(\frac{\gamma T_N}{2n} < \bar{r}_i \leq 3T_N; K = n\right) \end{aligned}$$

since $\bar{r}_i \leq 3T_N$ for $i \leq K - 1$. The aim is to prove that the right-hand side of the previous inequality can be made smaller than $\varepsilon/2$. First of all, (5.1) and (5.7) imply that

$$P_0^N(\Omega_N) \geq P_0^N(\Omega_N \mid \pi_N < \infty)P_0^N(\pi_N < \infty) \geq (1 - \varepsilon_N)P_0(\tau = \infty).$$

In particular, by letting $p = P_0(\tau = \infty)$ denote the survival probability of the contact process with parameter $\beta > \lambda_c$ and applying Lemma 5.3, we obtain

$$P^\eta(K = 1) = P_0^N(\tau_N > 3T_N) \geq P_0^N(\Omega_N) - P_0^N(\tau_N \leq 3T_N; \Omega_N) \geq p/2 \tag{5.13}$$

for all N sufficiently large. This, together with the fact that K is geometrically distributed (see (5.11) above), implies that for all N sufficiently large

$$\sum_{n=n_\varepsilon}^\infty \sum_{i=1}^{n-1} P^\eta(K = n) \leq \sum_{n=n_\varepsilon}^\infty (n-1) \left(1 - \frac{p}{2}\right)^{n-1}. \tag{5.14}$$

In particular, there exists a large enough n_ε , fixed from now on, such that the left hand side of (5.14) is smaller than $\varepsilon/4$. To deal with the first n_ε terms, we observe that, since $0 < b < c$,

$$\begin{aligned} \sum_{n=1}^{n_\varepsilon} \sum_{i=1}^{n-1} P^\eta\left(r_i > \frac{\gamma T_N}{2n}\right) &\leq n_\varepsilon^2 P^\eta\left(r_1 > \frac{\gamma T_N}{2n_\varepsilon}\right) \leq n_\varepsilon^2 \exp\left(-\frac{\gamma T_N B_N}{2n_\varepsilon}\right) \\ &\leq n_\varepsilon^2 \exp\left(-\frac{\gamma}{2n_\varepsilon} \exp((c-b)N^2)\right) \leq \frac{\varepsilon}{8} \end{aligned}$$

for N sufficiently large. In other respects, Lemma 5.4 implies that

$$\begin{aligned} \sum_{n=1}^{n_\varepsilon} \sum_{i=1}^{n-1} P^\eta\left(\frac{\gamma T_N}{2n} < \bar{r}_i \leq 3T_N\right) &\leq n_\varepsilon^2 P^\eta\left(\frac{\gamma T_N}{2n_\varepsilon} < \bar{r}_1 \leq 3T_N\right) \\ &\leq n_\varepsilon^2 P^\eta\left(\frac{\gamma T_N}{2n_\varepsilon} < \tau_N \leq 3T_N\right) \\ &\leq n_\varepsilon^2 \left[\varepsilon'_N + C_1 \exp\left(-\gamma_1 \frac{\gamma T_N}{2n_\varepsilon}\right)\right] \leq \frac{\varepsilon}{8} \end{aligned}$$

for N sufficiently large. Putting things together, we get

$$P^\eta(\sigma_{e_1} < T_N \mid (0, 0) \text{ is occupied}) \geq 1 - \varepsilon/2. \tag{5.15}$$

Now, equation (5.9) in Lemma 5.3 with $s = T_N - \sigma_{e_1}$ implies that

$$P^\eta(f_N(1, 1) \geq \gamma \mid \sigma_{e_1} < T_N) \geq 1 - \varepsilon/2 \tag{5.16}$$

for N sufficiently large. This, together with inequality (5.15), implies that site $(1, 1)$ is occupied with probability at least $1 - \varepsilon$ whenever site $(0, 0)$ is occupied. This completes the proof. \square

Lemma 5.1 is a straightforward consequence of Lemma 5.5. To construct our nontrivial stationary distribution, we start η_t from the “all 1” configuration. The attractivity of the process, together with (5.9) and (5.16), implies that for all $z \in 2\mathbb{Z}$ and all N sufficiently large

$$\begin{aligned} P^\eta((z, 0) \text{ is occupied}) \\ \geq P^\eta((z, 0) \text{ is occupied} \mid \sigma_{ze_1} = 0) P^\eta(\sigma_{ze_1} = 0) \geq 1/2 \times p/2 > 0. \end{aligned}$$

In particular, there are infinitely many $z \in 2\mathbb{Z}$ such that $(z, 0)$ is occupied. By using one more time the attractivity of the process, we obtain that η_t converges in distribution to its upper stationary measure that we denote by μ (see Theorem 2.7 in [7]). Finally, since percolation occurs with positive probability when $\varepsilon > 0$ is small, it follows from Lemma 5.1 that

$$\liminf_{n \rightarrow \infty} \mu((z, n) \text{ is occupied}) \geq \liminf_{n \rightarrow \infty} P_\varepsilon((z, n) \text{ is wet}) > 0 \quad \text{if } N \geq N_\varepsilon$$

where P_ε denotes the law of the percolation process with parameter $1 - \varepsilon$. This implies that μ concentrates on configurations with infinitely many 1's.

Acknowledgment

The authors would like to thank Rick Durrett, Roberto Fernández, and Ellen Saada for fruitful discussions, as well as two anonymous referees for their suggestions.

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