Flying in the wind

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An ill wind

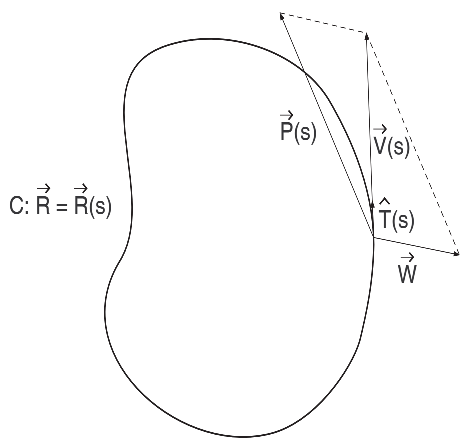
You've probably heard the expression "Any wind is an ill wind". When it comes to flying, that isn't true if the wind happens to be a tailwind for your trip. However, it is true if you are talking about flying a closed course. Specifically, if the wind has constant speed and direction and you fly a closed course at a constant airspeed, your best possible time around the course is when the wind is calm.

Description

We will assume that the course the airplane flies can be considered as a simple closed curve in the xy plane and that the speed of the airplane is greater than that of the wind. Otherwise, of course, the plane couldn't negotiate the course. To negotiate a course in the wind, the pilot must "crab" the airplane to account for the effects of the wind and remain on course. Part of the time the wind may be at his tail, increasing his speed over the ground, and part of the time he is heading into the wind, decreasing his ground speed. We want to show that the time taken to traverse the course is least when there is no wind.

Analysis

We will be using a bit of vector calculus in what follows.



Velocity vectors

Refer to the above figure. The curve C represents the closed course and is parameterized by arc length s:

$$C : \vec{R} = \vec{R}(s),\ 0 \le s \le \ell(C)$$

where $\ell(C)$ represents the length of C. The velocity vector for the airplane is $\vec{P}(s)$ with constant length $p = |\vec{P}(s)|$ representing the airplane's speed. The wind is represented by the constant vector $\vec{W}$ with length $w = |\vec{W}|$ representing the speed of the wind. Note that while the wind is represented by a constant vector, the airplane's velocity vector depends on its position s along the curve because its direction changes. Finally, the vector sum of the wind and airplane vectors gives the ground velocity vector $\vec{V}(s)$ whose length $v(s) = |\vec{V}(s)|$ represents the speed along the ground. We need to compute the time to traverse the course in the wind, which is given by

$$t_{wind} = \int_0^{\ell(C)} \frac{1}{v(s)}\, ds$$

and show that this time is always at least as long as the time it would take to traverse the curve C with no wind at the constant speed p of the airplane, which is

$$t_{no\ wind} = \frac{\ell(C)}{p}$$

The pilot of the airplane continuously adjusts his heading so that the ground velocity vector is tangent to the curve C. If we let $\hat{T}(s)$ be the unit vector tangent to the curve, then the ground velocity is

$$\vec{V}(s) = v(s)\hat{T}(s)$$

We calculate the ground velocity by adding the wind and airplane velocities:

$$\vec{P}(s) + \vec{W} = \vec{V}(s) = v(s)\hat{T}(s)$$

Solving this equation for $\vec{P}(s)$ gives

$$\vec{P}(s) = v(s)\hat{T}(s) - \vec{W}$$

To calculate $v(s)$ we take the dot product of this equation with itself

$$|\vec{P}(s)|^2 = v^2(s) - 2v(s)\vec{W} \cdot \hat{T}(s) + |\vec{W}|^2 = v^2(s) - 2v(s)\vec{W} \cdot \hat{T}(s) + w^2$$

which gives a quadratic equation for $v(s)$.

$$v^2(s) - 2v(s)\vec{W} \cdot \hat{T}(s) + (w^2 - p^2) = 0$$

The quadratic formula yields

$$v(s) = \vec{W} \cdot \hat{T}(s) \pm \sqrt{(\vec{W} \cdot \hat{T}(s))^2 + (p^2 - w^2)}$$

from which we may drop the minus choice which would give a negative value for $v(s)$.

The equation for t_{wind} becomes

$$\begin{aligned} t_{wind} &= \int_0^{\ell(C)} \frac{1}{v(s)}\, ds \\ &= \int_0^{\ell(C)} \frac{1}{\vec{W}\cdot\hat{T}(s) + \sqrt{(\vec{W}\cdot\hat{T}(s))^2 + (p^2 - w^2)}}\, ds \\ &= \int_0^{\ell(C)} \frac{\vec{W}\cdot\hat{T}(s) - \sqrt{(\vec{W}\cdot\hat{T}(s))^2 + (p^2 - w^2)}}{(\vec{W}\cdot\hat{T}(s))^2 - ((\vec{W}\cdot\hat{T}(s))^2 + (p^2 - w^2))}\, ds \\ &= \int_0^{\ell(C)} \frac{\vec{W}\cdot\hat{T}(s) - \sqrt{(\vec{W}\cdot\hat{T}(s))^2 + (p^2 - w^2)}}{(w^2 - p^2)}\, ds \\ &= \int_0^{\ell(C)} \frac{\vec{W}\cdot\hat{T}(s)}{(w^2 - p^2)}\, ds + \int_0^{\ell(C)} \frac{\sqrt{(\vec{W}\cdot\hat{T}(s))^2 + (p^2 - w^2)}}{(p^2 - w^2)}\, ds \\ &\geq \frac{1}{w^2 - p^2} \oint_C \vec{W}\cdot d\vec{R} + \frac{1}{p^2 - w^2} \int_0^{\ell(C)} \sqrt{p^2 - w^2}\, ds \end{aligned}$$

The line integral in the first term is 0 because the constant vector field $\vec{W}$ is conservative. So we have

$$\begin{aligned} t_{wind} &\geq \frac{1}{\sqrt{p^2 - w^2}} \int_0^{\ell(C)} 1\, ds \\ &= \frac{\ell(C)}{\sqrt{p^2 - w^2}} \geq \frac{\ell(C)}{p} = t_{no\ wind} \end{aligned}$$

which completes the argument.