Homework Set 9

Extra #1

Steady State eqn $\quad -\Delta u = +\frac{1}{k}F(x,y)$

First solve the eigenvalue problem $\quad -\Delta\phi = \lambda\phi$

$$\Rightarrow \phi = X(x)Y(y) \qquad -\frac{X''}{X} = \frac{Y''}{Y} + \lambda$$

$\therefore -X'' = c_1 X \qquad X(0)=0 \quad X(a)=0$

$\therefore X_m = \sin\frac{m\pi x}{a} \qquad m=1,2,\ldots$

$-Y'' = c_2 Y \qquad Y'(0)=0 \qquad Y(b)=0$

$\Big\}$ Homo BC!

We easily see that $c_2 > 0$ ~~...~~ $\therefore Y = A\cos\mu y + B\sin\mu y$

$Y'(0) = \mu B = 0 \;\therefore B = 0 \quad \therefore Y = \cos\mu y$

$Y(b) = \cos\mu b = 0 \quad \therefore \mu b = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

~~...~~ $\mu_n = \dfrac{(2n-1)\pi}{2b}$

$$\phi_{mn} = \sin\frac{m\pi x}{a}\cos\frac{(2n-1)\pi}{2b}$$

$$\lambda_{mn} = \frac{m^2\pi^2}{a^2} + \frac{(2n-1)^2\pi^2}{4b^2}$$

$-\Delta v^{(1)} = \frac{1}{k}F \quad$ in $0<x<a,\ 0<y<b$ with homogeneous BC.

$$v^{(1)} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{mn}\phi_{mn}$$

$$F = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} F_{mn}\phi_{mn} \quad \text{where, by orthogonality}$$

$$F_{mn} = \frac{\int_0^b\int_0^a F(x,y)\phi_{mn}(x,y)\,dx\,dy}{\int_0^b\int_0^a \phi_{mn}(x,y)^2\,dx\,dy}$$

$$\therefore F_{mn} = \frac{4}{ab}\int_0^b\int_0^a F(x,y)\sin\frac{m\pi x}{a}\cos\frac{(2n-1)\pi y}{2b}$$

pluggin we note $\quad V_{mn} = F_{mn}/(\lambda_{mn}k)$

Next we solve $-\Delta v^{(2)} = 0$ with nonhomo b.c.

$v_y^{(2)}(x,0) = 0 \quad v^{(2)}(x,b) = 0 \quad v^{(2)}(0,y) = 0 \quad v^{(2)}(a,y) = \phi(y)$

Separate variables:

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

$$+\frac{X''}{X} = -\frac{Y''}{Y} = c \qquad X(0) = 0$$

$$Y'(0) = 0 \quad Y(b) = 0$$

B.C. on Y are homogeneous $\therefore$ solve

$$-Y'' = cY \quad Y'(0) = 0 \quad Y(b) = 0$$

$\therefore$ get $\quad Y_m = \cos\frac{(2m-1)\pi y}{2b} \qquad c = \frac{(2m-1)^2\pi^2}{4b^2}$

$$\therefore \quad X'' = \frac{(2m-1)^2\pi^2}{4b^2} X$$

$$\therefore X = A\cosh\frac{(2m-1)\pi x}{2b} + B\sinh\frac{(2m-1)\pi x}{2b}$$

B.C. $X(0) = 0 \Rightarrow A = 0 \therefore X_m = \sinh\frac{(2m-1)\pi x}{2b}$

$$v^{(2)} = \sum_{m=1}^{\infty} \alpha_m \sinh\frac{(2m-1)\pi x}{2b}\cos\frac{(2m-1)\pi y}{2b}$$

Set $x = a$

$$\phi(y) = \sum \alpha_m \sinh\frac{(2m-1)\pi a}{2b}\cos\frac{(2m-1)\pi y}{2b}$$

$\therefore$ by orthogonality : $\alpha_m \sinh\frac{(2m-1)\pi a}{2b} = \dfrac{\int_0^b \phi(y)\cos\frac{(2m-1)\pi y}{2b}\,dy}{\int_0^b \cos^2\frac{(2m-1)\pi y}{2b}\,dy}$

$$\therefore \alpha_m = \frac{2}{b\sinh\frac{(2m-1)\pi a}{2b}}\int_0^b \phi(y)\cos\frac{(2m-1)\pi y}{2b}\,dy$$

Let $u = v + w$ where $v = v^{(1)} + v^{(2)}$
Then w is transient ~~&~~ solution

$$w_t = k\Delta w \quad \text{in } 0<x<a,\ 0<y<b$$
$$w(x,y,0) = u_0(x,y) - v^{(1)}(x,y) - v^{(2)}(x,y)$$

Let $w_0 = u_0 - v^{(1)} - v^{(2)}$ $\therefore w(x,y,0) = w_0(x,y)$
w satisfies homo. b.c.

Let $w = T(t)\,W(x,y)$

$$\frac{T'}{kT} = \frac{\Delta W}{W} = \text{cst.}$$

$\therefore$ $\Delta W = \text{cst}\, W$ + homo B.C.
Already solved: $W = \phi_{mn}$ $\quad \text{cst} = -\lambda_{mn}$
$\therefore$ $T = c\,e^{-\lambda_{mn} k t}$

$\therefore$
$$w(x,y,t) = \sum_{\substack{m=1\\n=1}}^{\infty} c_{mn}\,\phi_{mn}(x,y)\,e^{-\lambda_{mn} k t}$$

$$c_{mn} = \frac{4}{ab}\int_0^b\int_0^a w_0(x,y)\,\phi_{mn}(x,y)\,dx\,dy$$

Final Answer

(a) steady state solution $v := v^{(1)} + v^{(2)}$

$$v^{(1)}(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{F_{mn}}{\lambda_{mn} k}\, \phi_{mn}(x,y)$$

where $\phi_{mn}(x,y) = \sin\frac{m\pi x}{a}\cos\frac{(2n-1)\pi y}{2b}$

$$F_{mn} = \frac{4}{ab}\int_0^b\int_0^a F(x,y)\,\phi_{mn}(x,y)\,dx\,dy$$

$$\lambda_{mn} = \frac{m^2\pi^2}{a^2} + \frac{(2n-1)^2\pi^2}{4b^2}$$

$$v^{(2)}(x,y) = \sum_{n=1}^{\infty} \alpha_n \sinh\frac{(2n-1)\pi x}{2b}\cos\frac{(2n-1)\pi y}{2b}$$

where $$\alpha_n = \frac{2\int_0^b \phi(y)\cos\frac{(2n-1)\pi y}{2b}\,dy}{b\sinh\frac{(2n-1)\pi a}{2b}}$$

(b)

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\,\phi_{mn}(x,y)\,e^{-\lambda_{mn} k t}$$

where

$$c_{mn} = \frac{4}{ab}\int_0^b\int_0^a w_0(x,y)\,\phi_{mn}(x,y)\,dx\,dy$$

and $w_0(x,y) = u_0(x,y) - v^{(1)}(x,y) - v^{(2)}(x,y)$