Solution of set 8, #3.

Let $\Omega = \frac{\omega^2}{c^2}$ $\quad T = \frac{2\alpha\omega}{c^2}$

$$\begin{pmatrix} u_1'' \\ u_2'' \end{pmatrix} = \begin{pmatrix} -\Omega & T \\ -T & -\Omega \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \therefore \quad \vec{u}'' = M\vec{u}$$

Find eigenvalues: $\det\begin{pmatrix} -\Omega-\lambda & T \\ -T & -\Omega-\lambda \end{pmatrix} = (\lambda+\Omega)^2 + T^2 = 0$

$\therefore \lambda = -\Omega \pm iT \quad \Rightarrow \quad M - \lambda I = \begin{pmatrix} \mp iT & T \\ -T & \mp iT \end{pmatrix}$

which obviously has eigenvectors $\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$

$$P := \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix}$$

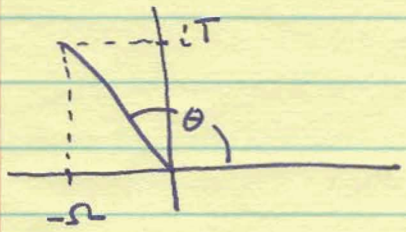
an $P^{-1}MP = \begin{pmatrix} -\Omega + iT & 0 \\ 0 & -\Omega - iT \end{pmatrix}$

$$\therefore \quad M = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} -\Omega+iT & 0 \\ 0 & -\Omega-iT \end{pmatrix}\begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix}$$

Let $\vec{v} = P^{-1}\vec{u} \quad \therefore \vec{u} = P\vec{v}$ then

$$v'' = \begin{pmatrix} -\Omega + iT & 0 \\ 0 & -\Omega - iT \end{pmatrix}\vec{v}$$

$\therefore \quad v_1'' = (-\Omega + iT)v_1 \qquad v_2'' = (-\Omega - iT)v_2$



Let $\theta = \text{Arg}(-\Omega + iT$

$\rho = (\Omega^2 + T^2)^{1/4}$ so that

$$-\Omega + iT = \rho^2 e^{2i\theta}$$
$$-\Omega - iT = \rho^2 e^{-2i\theta}$$

$(-\Omega + iT)^{1/2} = \pm\rho e^{i\theta} \qquad$ Let $\rho e^{i\theta} = \alpha + i\beta$

$(-\Omega - iT)^{1/2} = \pm\rho e^{-i\theta} \qquad \therefore \alpha = \rho\cos\theta \quad \beta = \rho\sin\theta$

Then $v_1 = E e^{(\alpha+i\beta)x} + B e^{-(\alpha+i\beta)x}$

$v_2 = C e^{(\alpha-i\beta)x} + D e^{-(\alpha-i\beta)x}$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} E e^{(\alpha+i\beta)x} + B e^{-(\alpha+i\beta)x} \\ C e^{(\alpha-i\beta)x} + D e^{-(\alpha-i\beta)x} \end{pmatrix}$$

$$u_1 = e^{\alpha x}\left[E e^{i\beta x} + C e^{-i\beta x}\right] + e^{-\alpha x}\left[B e^{-i\beta x} + D e^{i\beta x}\right]$$
$$u_2 = i e^{\alpha x}\left[E e^{i\beta x} - C e^{-i\beta x}\right] + i e^{-\alpha x}\left[B e^{-i\beta x} - D e^{i\beta x}\right]$$

$$\left.\begin{aligned} u_1(0) = 0 &\Rightarrow E + C + B + D = 0 \\ u_2(0) = 0 &\Rightarrow E - C + B - D = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} E + B = 0 \\ C + D = 0 \end{aligned}$$

$\therefore$

$$u_1 = e^{\alpha x}\left[E e^{i\beta x} + C e^{-i\beta x}\right] - e^{-\alpha x}\left[E e^{-i\beta x} + C e^{i\beta x}\right]$$
$$u_2 = i e^{\alpha x}\left[E e^{i\beta x} - C e^{-i\beta x}\right] - i e^{-\alpha x}\left[E e^{-i\beta x} - C e^{i\beta x}\right]$$

$$u_1(L) = e^{\alpha L}\left[E e^{i\beta L} + C e^{-i\beta L}\right] - e^{-\alpha L}\left[E e^{-i\beta L} + C e^{i\beta L}\right] = 0$$

$$\Rightarrow \quad \frac{E}{C} = \frac{e^{-\alpha L + i\beta L} - e^{\alpha L - i\beta L}}{e^{\alpha L + i\beta L} - e^{-i\beta L - \alpha L}} = -\frac{\gamma}{\bar{\gamma}}$$

where $\gamma = e^{-\alpha L + i\beta L} - e^{\alpha L - i\beta L}$

$$u_2(L) = \left(i e^{\alpha L + i\beta L} - i e^{-\alpha L - i\beta L}\right) E + \left(i e^{-\alpha L - i\beta L} - i e^{\alpha L - i\beta L}\right) C = A$$

$$\therefore \quad A = C i\gamma - E i\bar{\gamma} = 2i\gamma C$$

$$\therefore \quad C = \frac{A}{2i\gamma}, \quad E = -\frac{A}{2i\bar{\gamma}}, \quad D = -\frac{A}{2i\gamma}, \quad B = \frac{A}{2i\bar{\gamma}}$$

$$u_1 = \frac{A}{2i}\left[e^{+\alpha x}\left(-\frac{e^{i\beta x}}{\bar{\gamma}} + \frac{e^{-i\beta x}}{\cancel{\bar{\gamma}}\gamma}\right) + e^{-\alpha x}\left(\frac{e^{-i\beta x}}{\bar{\gamma}} - \frac{e^{i\beta x}}{\gamma}\right)\right.$$

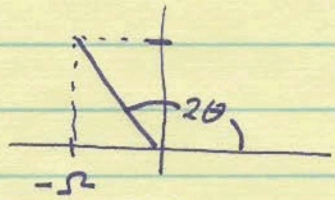
$$u_2 = \frac{A}{2}\left[e^{\alpha x}\left(-\frac{e^{i\beta x}}{\bar{\gamma}} - \frac{e^{-i\beta x}}{\gamma}\right) + e^{-\alpha x}\left(\frac{e^{-i\beta x}}{\bar{\gamma}} + \frac{e^{i\beta x}}{\gamma}\right)\right]$$

Finally let $\frac{1}{\gamma} = \sigma e^{i\phi}$ $\therefore$ $\frac{1}{\bar{\gamma}} = \sigma e^{-i\phi}$

$$u_1 = A\sigma\left(-\sin(\beta x - \phi)\, e^{\alpha x} - e^{-\alpha x}\sin(\beta x + \phi)\right)$$
$$u_2 = A\sigma\left(-e^{\alpha x}\cos(\beta x - \phi) + e^{-\alpha x}\cos(\beta x + \phi)\right)$$

Solution:

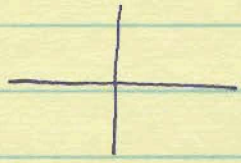
Let $\Omega := \dfrac{\omega^2}{c^2}$ $\qquad T := \dfrac{2\alpha\omega}{c^2}$



$\rho := (\Omega^2 + T^2)^{1/4}$ $\qquad \theta = \frac{1}{2}\mathrm{Arg}(-\Omega + iT)$

$\cancel{\alpha := \pm\rho\cos\alpha}$ $\qquad \alpha := \rho\cos\theta$ $\qquad \beta := \rho\sin\theta$

$\gamma := e^{-\alpha L + i\beta L} - e^{\alpha L - i\beta L}$



$\sigma := \dfrac{1}{|\gamma|}$ $\qquad \phi = -\mathrm{Arg}(\gamma)$

then

$$u_1 = -\sigma A\left[e^{\alpha x}\sin(\beta x - \phi) + e^{-\alpha x}\sin(\beta x + \phi)\right]$$
$$u_2 = \sigma A\left[-e^{\alpha x}\cos(\beta x - \phi) + e^{-\alpha x}\cos(\beta x + \phi)\right]$$