Set 8

1. a) $-(py')' + qy = \lambda r y \qquad \bar{y}(a)y'(a) \geq 0 \quad \bar{y}(b)y'(b) \leq 0$

$$-\int_a^b [\bar{y}(py')' + qy\bar{y}]\,dx = \lambda \int_a^b ry\bar{y}\,dx$$

$$-\bar{y}py'\Big|_a^b + \int_a^b [py'\bar{y}' + qy\bar{y}]\,dx = \lambda \int_a^b ry\bar{y}\,dx$$

$$-\bar{y}(b)p(b)y'(b) + \bar{y}(a)p(a)y'(a) + \int_a^b p|y'|^2\,dx + \int_a^b \min(q(x))|y|^2\,dx \leq \lambda \int_a^b \max(r(x))|y|^2\,dx$$

$$\therefore \min(q(x)) \int_a^b |y|^2\,dx \leq \lambda \max(r(x)) \int_a^b |y|^2\,dx$$

$$\therefore \lambda \geq \frac{\min(q(x))}{\max(r(x))}$$

b) If we have equality ~~so~~ then we need

$\bar{y}(b)y'(b) = 0 \quad \bar{y}(a)y'(a) = 0 \quad \int_a^b p|y'|^2\,dx = 0$

$q(x) \equiv \min(q(x)) \qquad r(x) \equiv \max(r(x))$

$\therefore$ r and q are constant functions

$y \equiv$ constant (since $y' \equiv 0$)

(but $y \not\equiv 0$!) $\therefore \bar{y}(b) \neq 0$ and $\bar{y}(a) = 0$

$\therefore y'(b) = 0$ and $y'(a) = 0$

Set 8

2. Separating variables leads to $\frac{T'}{kT} = \frac{X''}{X} = \lambda^2$

$$X'' = \lambda^2 X \qquad X'(0) = -\frac{\alpha}{L} X(0) \qquad X'(L) = \frac{\beta}{L} X(L)$$

$\therefore$

$$X = Ae^{\lambda x} + Be^{-\lambda x} \qquad X' = \lambda A e^{\lambda x} - \lambda B e^{-\lambda x}$$

BC: $\lambda(A - B) = -\frac{\alpha}{L}(A + B)$

$\lambda(Ae^{\lambda L} - Be^{-\lambda L}) = \frac{\beta}{L}(Ae^{\lambda L} + Be^{-\lambda L})$ Let $s = \lambda L$

$\Rightarrow$

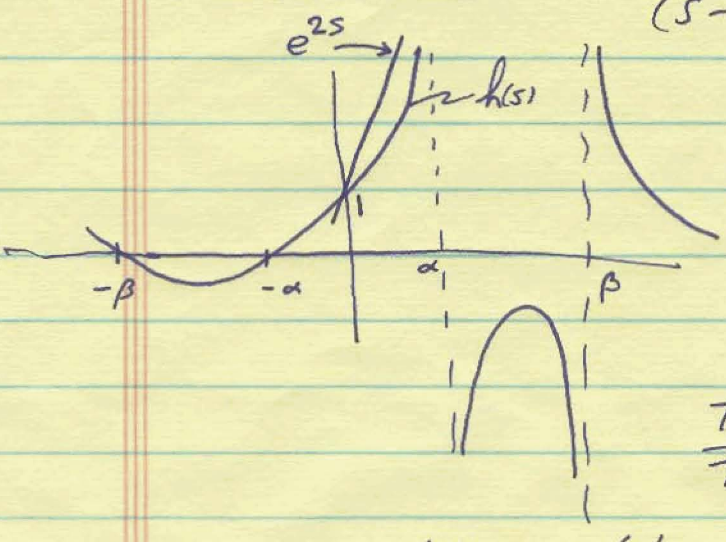
$$(s+\alpha)A + (\alpha - s)B = 0$$
$$e^{2s}(s-\beta)A - (s+\beta)B = 0$$

To get a nontrivial solution we need

$$\det\begin{pmatrix} s+\alpha & \alpha - s \\ e^{2s}(s-\beta) & -(s+\beta) \end{pmatrix} = 0$$

$$-(s+\alpha)(s+\beta) + (s-\alpha)e^{2s}(s-\beta) = 0$$

$$\therefore\ h(s) = e^{2s} - \frac{(s+\alpha)(s+\beta)}{(s-\alpha)(s-\beta)} = 0$$



Let $T(s) = \frac{(s+\alpha)(s+\beta)}{(s-\alpha)(s-\beta)}$

We need $T'(0)$

$\ln T = \ln(s+\alpha) + \ln(s+\beta) - \ln(s-\alpha) - \ln(s-\beta)$

$$\frac{T'}{T} = \frac{1}{s+\alpha} + \frac{1}{s+\beta} - \frac{1}{s-\alpha} - \frac{1}{s-\beta}$$

$$\frac{T'(0)}{T(0)} = 2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \qquad T(0) = 1$$

$$\therefore\ T'(0) = 2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \qquad \frac{d}{ds}e^{2s}\Big|_{s=0} = 2$$

$\therefore$ if $2 > T'(0)$ then $T(s)$ and e^{2s} must intersect between 0 and α and again between β and ∞

$\therefore$ Need $\frac{1}{\alpha} + \frac{1}{\beta} < 1$ for 2 solns.

3 Let $u = u_1 \cos\omega t + u_2 (\sin\omega t)$ (*)

Clearly we want $\boxed{u_1(0)=0 \quad u_2(0)=0 \quad u_1(L)=0 \quad u_2(L)=A}$

Plug (*) into PDE

$$-\omega^2 u_1(x)\cos\omega t - \omega^2 u_2(x)\sin\omega t - 2\alpha\omega\, u_1(x)\sin\omega t$$
$$+ 2\alpha\omega\, u_2(x)\cos\omega t = c^2\left(u_1''\cos\omega t + u_2''\sin\omega t\right)$$

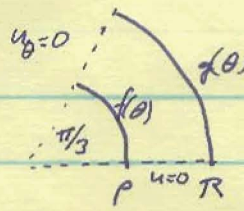
$\Rightarrow$

$$\boxed{\begin{aligned} u_1'' &= -\frac{\omega^2}{c^2}u_1 + \frac{2\alpha\omega}{c^2}u_2 \\ u_2'' &= -\frac{\omega^2}{c^2}u_2 - \frac{2\alpha\omega}{c^2}u_1 \end{aligned}}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}'' = \begin{pmatrix} -\frac{\omega^2}{c^2} & \frac{2\alpha\omega}{c^2} \\ -\frac{2\alpha\omega}{c^2} & -\frac{\omega^2}{c^2} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

One way to solve this is to substitute: Solve the 1st eqn for u_2 in terms of u_1 then substitute that into the 2nd eqn, yielding a 4th order ode for u_1. Another way is to use Laplace transforms to reduce it to an algebraic system which can be solved etc. A third method is to use some linear algebra to diagonalize the system (i.e. uncouple). I have solved this problem using the last method.

4.



$$\frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta} = 0 \qquad \text{try} \quad u = R(r)\,\Theta(\theta)$$

$$\frac{1}{r}(rR')'\Theta + \frac{1}{r^2} R\,\Theta'' = 0$$

$$\frac{r(rR')'}{R} = -\frac{\Theta''}{\Theta} = K$$

(As in class) $\quad -\Theta'' = K\Theta \qquad \Theta(0)=0 \quad \Theta'(\frac{\pi}{3})=0$

From problem 1 we know $K>0 \quad \therefore K=\lambda^2 \quad \lambda>0$

$$\Theta = A\cos\lambda x + B\sin\lambda x$$

$$\Theta(0) = A = 0 \quad \therefore \quad \Theta = \sin\lambda x$$

$$\Theta'(\tfrac{\pi}{3}) = \lambda\cos(\tfrac{\lambda\pi}{3}) = 0 \quad \therefore \quad \frac{\lambda\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$\lambda_m = \frac{3(2m-1)}{2}$$

$$\Theta_m = \sin(\lambda_m\theta)$$

$$r^2 R_m'' + r R_m' - \lambda_m^2 R_m = 0$$

try $R_m = r^p \qquad r^2 p(p-1) r^{p-2} + r\,p\,r^{p-1} - \lambda_m^2 r^p = 0$

$$p^2 - \lambda_m^2 = 0 \quad \therefore \quad p = \pm\lambda_m$$

$$R_m = A_m r^{\lambda_m} + B_m r^{-\lambda_m}$$

$$u = \sum_{m=1}^{\infty} \left(A_m r^{\lambda_m} + B_m r^{-\lambda_m}\right)\sin(\lambda_m\theta)$$

$$\sum_{m=1}^{\infty} \left(A_m \rho^{\lambda_m} + B_m \rho^{-\lambda_m}\right)\sin(\lambda_m\theta) = \sum_{m=1}^{\infty} F_m \sin(\lambda_m\theta)$$

$$\sum_{m=1}^{\infty} \left(A_m R^{\lambda_m} + B_m R^{-\lambda_m}\right)\sin(\lambda_m\theta) = \sum_{m=1}^{\infty} G_m \sin(\lambda_m\theta)$$

$$F_m = \int_0^{\pi/3} f(\theta)\sin\lambda_m\theta\,d\theta \Big/ \int_0^{\pi/3} \sin^2\lambda_m\theta\,d\theta$$

$$\int_0^{\pi/3} \sin^2 \lambda_m \theta \, d\theta = \int_0^{\pi/3} \frac{1 - \cos 2\lambda_m \theta}{2} \, d\theta$$

$$= \frac{\pi}{6} - \frac{1}{4\lambda_m} \sin(3(2m-1)\theta)\Big|_0^{\pi/3} = \frac{\pi}{6}$$

ANSWER

$$F_m = \frac{6}{\pi} \int_0^{\pi/3} f(\theta) \sin(\lambda_m \theta) \, d\theta$$ similarly

$$G_m = \frac{6}{\pi} \int_0^{\pi/3} g(\theta) \sin(\lambda_m \theta) \, d\theta$$

$$\rho^{\lambda_m} A_m + \rho^{-\lambda_m} B_m = F_m$$
$$R^{\lambda_m} A_m + R^{-\lambda_m} B_m = G_m$$

$$\therefore A_m = \frac{\begin{vmatrix} F_m & \rho^{-\lambda_m} \\ G_m & R^{-\lambda_m} \end{vmatrix}}{\begin{vmatrix} \rho^{\lambda_m} & \rho^{-\lambda_m} \\ R^{\lambda_m} & R^{-\lambda_m} \end{vmatrix}} = \frac{R^{-\lambda_m} F_m - \rho^{-\lambda_m} G_m}{(\rho/R)^{\lambda_m} - (R/\rho)^{\lambda_m}}$$

$$B_m = \frac{\begin{vmatrix} \rho^{\lambda_m} & F_m \\ R^{\lambda_m} & G_m \end{vmatrix}}{\begin{vmatrix} & \\ & \end{vmatrix}} = \frac{\rho^{\lambda_m} G_m - R^{\lambda_m} F_m}{(\rho/R)^{\lambda_m} - (R/\rho)^{\lambda_m}}$$

$$\therefore u = \sum_{m=1}^{\infty} \left(A_m r^{\lambda_m} + B_m r^{-\lambda_m} \right) \sin(\lambda_m \theta)$$ where

$$A_m = \frac{R^{\lambda_m} G_m - \rho^{\lambda_m} F_m}{R^{2\lambda_m} - \rho^{2\lambda_m}} \qquad B_m = \frac{(R^2 \rho)^{\lambda_m} F_m - (\rho^2 R)^{\lambda_m} G_m}{R^{2\lambda_m} - \rho^{2\lambda_m}}$$