5.20 So f is continuously differentiable on (a_{i-1}, a_i) and has jumps $[[f(a_i)]] = f(a_i+0) - f(a_i-0)$ at a_i, $i \in J \subset \mathbb{Z}$. Define $[f'](x) = f'(x)$ for $a_{i-1} < x < a_i$. The value assigned at a_i's is inconsequential. Claim: $Df = [f'] + \sum_{i\in J} [[f(a_i)]]\delta_{a_i}$

Proof:

$$(Df, \phi) = -(f, D\phi) = -\sum_J \int_{a_{i-1}}^{a_i} f(x)\phi'(x)\,dx$$

$$\bigstar \quad = \sum_J \left[\int_{a_{i-1}}^{a_i} f'(x)\phi(x)\,dx + f(a_i)\phi(a_i) - f(a_{i-1})\phi(a_{i-1})\right]$$

More precisely, the last term is $f(a_i-0)\phi(a_i) - f(a_{i-1}+0) \times \phi(a_{i-1})$ and so their sum telescopes to $\sum_J [[f(a_i)]]\phi(a_i)$ $= \left(\sum_J [[f(a_i)]]\delta_{a_i}, \phi\right)$. Hence, by $\bigstar$,

$$(Df, \phi) = \left([f'] + \sum_J [[f(a_i)]]\delta_{a_i}, \phi\right)$$

5.22 $(u_{xx} - u_{tt}, \phi) = \iint_{\mathbb{R}^2} f(x+t)(\phi_{xx} - \phi_{tt})\,dx\,dt$

Change variables $\xi = x+t$, $\eta = x-t$ $\therefore \frac{\partial}{\partial x} = \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}$ and $\frac{\partial}{\partial t} = -\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}$ so that $\phi_{xx} - \phi_{tt} = 4\phi_{\xi\eta}$ and

$$dx\,dt = \left|\frac{\partial(x,t)}{\partial(\xi,\eta)}\right| d\xi\,d\eta \quad \left(\frac{\partial(x,t)}{\partial(\xi,\eta)}\right)^{-1} = \frac{\partial(\xi,\eta)}{\partial(x,t)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

$$\therefore \quad (u_{xx} - u_{tt}, \phi) = 4\iint_{\mathbb{R}^2} f(\xi)\,\tfrac{1}{2}\phi_{\xi\eta}\,d\xi\,d\eta$$

$$= 2\int_{-\infty}^{\infty} f(\xi)\left[\int_{-\infty}^{\infty} \phi_{\xi\eta}\,d\eta\right] d\xi = 2\int_{-\infty}^{\infty} f(\xi)\, 0 \,d\xi = 0 \quad \forall \phi \in \mathcal{D}(\mathbb{R})$$

$$\therefore \quad u_{xx} - u_{tt} = 0$$

5.24 Let $\phi_n \to 0$ in $\mathcal{S}(\mathbb{R})$. This implies in particular that $(1+x^2)\phi_n' \to 0$ uniformly on $\mathbb{R}$

$$\left| \int_{-\infty}^{\infty} e^x \cos(e^x)\, \phi_n \, dx \right| = \left| \int_{-\infty}^{\infty} \frac{d}{dx}[\sin(e^x)]\, \phi_n \, dx \right|$$

$$= \left| - \int_{-\infty}^{\infty} \sin(e^x)\, \phi_n'(x)\, dx \right| = \left| \int_{-\infty}^{\infty} \frac{\sin(e^x)}{1+x^2} \left[(1+x^2)\phi_n'\right] dx \right|$$

$$\leq \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, \|(1+x^2)\phi_n'\|_\infty \, dx \longrightarrow 0$$

$\Box$

5.28 $xy' = 0$. First we ~~show~~ solve $xf = 0$, $f \in \mathcal{D}'(\mathbb{R})$
Let $\phi_0 \in \mathcal{D}(\mathbb{R})$ such that $\phi_0(0) = 1$ i.e. pick one such test function. Now every other test function ϕ can be written as $\phi = \phi(0)\phi_0 + \psi$ where $\psi = \phi - \phi(0)\phi_0$ $\therefore \psi(0) = 0$ and hence $\psi(x)/x$ is another test function (after removing the singularity at 0). Hence if $\varphi \in \mathcal{D}(\mathbb{R})$ then $\exists\ \theta \in \mathcal{D}(\mathbb{R})$ s.t.

$$\varphi(x) = \varphi(0)\varphi_0(x) + x\theta(x)$$

If $xy' = 0$ then $(y', \varphi) = (y', \varphi(0)\varphi_0)$

$$+ (y', x\theta) = -(y, \varphi_0')(\delta, \phi) + (xy', \theta)$$

$$= (-c\delta, \phi) \quad \text{where } c = (y, \phi_0')$$

$$\therefore ((y + cH)', \phi) = 0 \quad \forall \phi \in \mathcal{D}(\mathbb{R})$$

$$\therefore y + cH = c_1 \quad \text{for some constant } c_1$$

Let $c_2 = -c$

$$y = c_1 + c_2 H$$

5.29. Let $\{h_i\}$ be a sequence of nonzero numbers converging to zero. Then it is easily seen that if $\phi \in \mathcal{D}(\mathbb{R})$ then $\frac{\phi(x-h_i) - \phi(x)}{h_i} \to -\phi'$ in $\mathcal{D}(\mathbb{R})$ $\therefore$

$$0 = \left(\frac{f(x+h_i) - f(x)}{h_i}, \phi \right) = \left(f, \frac{\phi(x-h_i) - \phi(x)}{h_i} \right)$$

$$\to (f, -\phi') = (Df^{\#}, \phi) \quad \therefore \quad Df = 0 \quad \therefore$$

$f = cst.$

5.30 Let $\phi_0 \in \mathcal{D}(\mathbb{R})$ s.t. $\int \phi_0 \, dx = 1$

We know $\forall \phi \in \mathcal{D}(\mathbb{R}) \quad \exists \psi \in \mathcal{D}(\mathbb{R})$ such that $\phi = \left[\int_{\mathbb{R}} \phi \, dx\right] \phi_0 + \psi'$

$$\left(\psi(x) = \int_{-\infty}^{x} \left[\phi(s) - \left[\int_{\mathbb{R}} \phi \, dx\right] \phi_0(s) \right] ds \right)$$

$$(F_n, \phi) = \left[\int_{\mathbb{R}} \phi \, dx\right] (F_n, \phi_0) + (F_n, \psi')$$

WLOG pick subsequence s.t. $(F_{n_k}, \phi_0) \to c$

$$(F_{n_k}, \phi) = \left[\int_{\mathbb{R}} \phi \, dx\right] (F_{n_k}, \phi_0) - (f_n, \psi)$$

$$\to (c, \phi) - (f, \psi)$$

5.40 $\Delta\Delta G = \delta$

$$\Delta G = \begin{cases} -r^{2-m}/(m-2)\Omega_m \equiv -k_m r^{2-m} & \text{if } m>2 \\ \frac{1}{2\pi}\ln r \equiv k_2 \ln r & \text{if } m=2 \end{cases}$$

$$\Delta = \frac{1}{r^{m-1}}\frac{d}{dr} r^{m-1}\frac{d}{dr}$$

Solving $\frac{1}{r^{m-1}}\frac{d}{dr} r^{m-1}\frac{dG}{dr} = \begin{cases} -k_m r^{2-m} & m>2 \\ k_2 \ln r & m=2 \end{cases}$

we get

$$G = \begin{cases} -\frac{1}{2}k_m r & m=3 \\ -\frac{1}{2}k_m \ln r & m=4 \\ \frac{1}{2(m-r)}k_m r^{4-m} & m>4 \\ \frac{r^2\ln r - r^2}{8\pi} & m=2 \end{cases}$$

5.51 $\Delta G = \delta \qquad \mathcal{F}(\Delta G) = \mathcal{F}(\delta)$

$-(\xi_1^2 + \xi_2^2 + \xi_3^2)\,\hat{G} = (2\pi)^{-3/2}$ Let $\rho = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$

$\hat{G} = -\dfrac{1}{(2\pi)^{3/2}\rho^2} \quad \therefore G = \dfrac{1}{(2\pi)^{3/2}} \iiint \hat{G}(\xi) e^{i\vec{\rho}\cdot\vec{r}}\, d\xi$

$= -\dfrac{1}{(2\pi)^3} \int_0^\infty \int_0^{2\pi} \int_0^\pi \dfrac{1}{\rho^2} e^{i\rho r\cos\theta} \rho^2 \sin\theta \, d\theta \, d\phi \, d\rho$

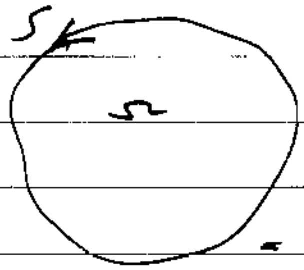
$= -\dfrac{1}{(2\pi)^2} \int_0^\infty \int_{-1}^1 e^{i\rho r\tau} \, d\tau \, d\rho$

$= -\dfrac{1}{(2\pi)^2} \int_0^\infty \dfrac{e^{i\rho r} - e^{-i\rho r}}{i\rho r} \, d\rho$

$= -\dfrac{1}{(2\pi)^2} \int_0^\infty 2 \dfrac{\sin \rho r}{\rho r} \, d\rho = \dfrac{-1}{\pi^2 r} \int_0^\infty \dfrac{\sin\sigma}{\sigma} \, d\sigma$

$= -\dfrac{1}{\pi^2 r} \cdot \dfrac{\pi}{2} = -\dfrac{1}{4\pi r}$

5.58



$$\int_\Omega [(\Delta\Delta u) v - u(\Delta\Delta v)]\,dA$$

$$= \int_\Omega [(\Delta\Delta u) v - \Delta u\,\Delta v]\,dA$$

$$+ \int_\Omega [\Delta u\,\Delta v - u(\Delta\Delta v)]\,dA$$

$$= \int_{\partial\Omega} \left[\left(\frac{\partial}{\partial n}\Delta u\right) v - \Delta u \frac{\partial v}{\partial n}\right] dl + \int_{\partial\Omega}\left(\frac{\partial u}{\partial n}\Delta v - u\frac{\partial}{\partial n}\Delta v\right) dl$$

$$\star = \int_{\partial\Omega} v \frac{\partial}{\partial n}\Delta u\; dl + \int_{\partial\Omega}\left[\left(\frac{\partial u}{\partial n} + \frac{\partial u}{\partial s}\right)\Delta v + u\frac{\partial}{\partial s}\Delta v\right] dl$$

$$- \int_{\partial\Omega} u \frac{\partial}{\partial n}\Delta v\; dl$$ where we used

$$\int_{\partial\Omega} \frac{\partial}{\partial s}(u\Delta v) = 0 \quad \text{(fundamental theorem of Calc)}$$

$$\star = \int_{\partial\Omega} v\left(\frac{\partial}{\partial n}\Delta u\right) dl - \int_{\partial\Omega} u\left(\frac{\partial}{\partial n} - \frac{\partial}{\partial s}\right)\Delta v\; dl$$

This is zero if $v = 0$ on $\partial\Omega$

$$\left(\frac{\partial}{\partial n} - \frac{\partial}{\partial s}\right)\Delta v = 0 \text{ on } \partial\Omega$$

$\therefore$ Adjoint operator of $\Delta\Delta$ is $\Delta\Delta$

the adjoint boundary conditions of $\{\Delta u = 0,\ \frac{\partial u}{\partial n} + \frac{\partial u}{\partial s} = 0\}$ are $\{v = 0,\ \frac{\partial \Delta v}{\partial n} - \frac{\partial \Delta v}{\partial s} = 0\}$

1a. $((\psi F)', \phi) = -(\psi F, \phi') = (F, -\psi\phi') =$
$(F, -(\psi\phi)' + \phi\psi') = (F', \psi\phi) + (\psi' F, \phi)$
$= (\psi F' + \psi' F, \phi)$

1b. in book.

2. Method: direct integration $\Delta u = 0$ if $r<1$ and $\Delta u = 0$ if $r>0$ $\Delta u = \frac{1}{r^2}(r^2 u_r)_r$
$\Rightarrow u = \frac{c_1}{r} + c_2$. We try to keep a continuity in u (but of course need a discontinuity in u' $(= u_r)$. No unique soln. of course. But it is nice to let $u \to 0$ as $r \to \infty$. Also we can't have a singularity at $r=0$ $\therefore$

$$u = \begin{cases} \frac{c}{r} & r \geq 1 \\ c & 0 \leq r \leq 1 \end{cases}$$

It remains to determine c:

$$(\Delta u, \phi) = \int_{|x|=1} \phi \, dx$$

Let's take ϕ radially symmetric $\therefore \phi = \phi(r)$

$\therefore 4\pi\,\phi(1) = (\Delta u, \phi) = (u, \Delta\phi)$. $\quad dV = 4\pi r^2 dr$

$$\therefore 4\pi\,\phi(1) = \int_0^1 c\,\frac{1}{r^2}(r^2\phi')'\,4\pi r^2 dr + \int_1^\infty \frac{c}{r}\cdot\frac{1}{r^2}(r^2\phi')'\,4\pi r^2 dr$$

$$= 4\pi c\left[r^2\phi'\right]_0^1 + 4\pi c\int_1^\infty \frac{1}{r}(r^2\phi')'\,dr = 4\pi c\,\phi'(1)$$

$$+ 4\pi c\left[\frac{1}{r}\,r^2\phi'\right]_1^\infty + 4\pi c\int_1^\infty \frac{1}{r^2}\,r^2\phi'\,dr = -4\pi c\,\phi(1)$$

$\therefore\ c = -1 \qquad\qquad u = \begin{cases} -\frac{1}{r} & r \geq 1 \\ -1 & 0 \leq r \leq 1 \end{cases}$

Method 2 (by Fourier Transforms)

From example 5.73 p. 156 $\quad \hat{F} = \sqrt{\frac{2}{\pi}}\ \frac{\sin\rho}{\rho}$

$\therefore\ \Delta u = F \quad \Rightarrow \quad -\rho^2 \hat{u} = \sqrt{\frac{2}{\pi}}\ \frac{\sin\rho}{\rho}$

$$\therefore\ \hat{u} = -\sqrt{\frac{2}{\pi}}\ \frac{\sin\rho}{\rho^3}$$

$$u = \frac{1}{(2\pi)^{3/2}}\left(-\sqrt{\frac{2}{\pi}}\right)\int_0^\infty\int_0^{2\pi}\int_0^\pi e^{i\rho r\cos\theta}\ \frac{\sin\rho}{\rho^3}\ \rho^2\sin\theta\ d\theta\, d\phi\, d\rho$$

$$= -\frac{1}{\pi}\int_0^\infty\int_{-1}^1 e^{i\rho r\tau}\ \frac{\sin\rho}{\rho}\ d\tau\ d\rho$$

$$= -\frac{1}{\pi}\int_0^\infty \frac{\sin\rho}{\rho}\ \frac{e^{i\rho r} - e^{-i\rho r}}{i\rho r}\ d\rho$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{\sin\rho}{\rho}\ \frac{\sin\rho r}{\rho r}\ d\rho$$

$$= \frac{1}{2\pi r}\int_0^\infty \frac{1}{\rho^2}\Big[\cos[(r+1)\rho] - \cos[(r-1)\rho]\Big]\, d\rho$$

$$= \begin{cases} -\frac{1}{r} & r > 0 \\ -1 & r < 0 \end{cases}$$

using Maple or integral tables