MAT 576
Fall 2008
Homework set 4 Answers

1. In class we looked at the wave equation $u_{xx} - u_{yy} = 0$ on a bounded domain $\Omega \subset \mathbb{R}^2$ that is partitioned into two (open) subdomains Ω_1 to the left of a smooth curve $\mathcal{C} : x = s(t)$ and Ω_2 to the right. Letting $[[w]] := w|_{\Omega_2} - w|_{\Omega_1}$ we saw that a function u that satisfies $u_{xx} - u_{yy} = 0$ in $\Omega_1 \cup \Omega_2$ is a weak solution only if for all $\phi \in C_0^\infty(\Omega)$ we have

$$\int_{\mathcal{C}} ([[u_x]]\, N_1\phi - [[u_y]]\, N_2\phi)\, d\ell + \int_{\mathcal{C}} ([[u]]\, N_1\phi_x - [[u]]\, N_2\phi_y)\, d\ell = 0,$$

where $\langle N_1, N_2 \rangle$ is the (continuously varying) unit normal to $\mathcal{C}$. Show that the function $H(x-y)$ satisfies this condition.

2. This is an exercise in using Green's theorem (also known as Gauss's theorem or Stokes's theorem, or Ostrogradskii's theorem or by various combinations of these names):

$$\int_\Omega u_{x_i} v\, dx = -\int_\Omega u v_{x_i}\, dx + \int_{\partial\Omega} uvN_i\, dS,$$

where $N := \langle N_1, \cdots, N_n \rangle$ is the unit outward normal to $\partial\Omega$ and Ω a bounded domain in $\mathbb{R}^n$, $n > 1$, with piecewise smooth boundary. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and suppose that u is a solution to the nonlinear elliptic problem

$$\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Suppose that f is continuous and define $F(s) = \int_0^s f(\sigma)\, d\sigma$. We employ the summation convention.

(a) Show that

$$\int_\Omega x_j \frac{\partial^2 u}{\partial x_j \partial x_i} \frac{\partial u}{\partial x_i}\, dx = -\frac{n}{2} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i}\, dx + \frac{1}{2} \int_{\partial\Omega} (N \cdot x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i}\, dS.$$

(b) Show that

$$\int_\Omega f(u) x_i \frac{\partial u}{\partial x_i}\, dx = \int_{\partial\Omega} F(u) x_i N_i\, dS - n \int_\Omega F(u)\, dx.$$

(c) Show that

$$\frac{n-2}{2} \int_\Omega \|\nabla u|^2\, dx + \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial N}\right)^2 (x \cdot N)\, dS + n \int_\Omega F(u)\, dx\, dx = 0.$$

Here $\partial u / \partial N = \nabla u \cdot N$ is the outward normal derivative. Hint: use the fact that $u = 0$ on the boundary - at least twice!

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Let Ω be a bounded domain in $\mathbb{R}^n$, $n > 1$, with piecewise smooth boundary. Let $N := (N_1, ..., N_n)^T$ be the unit outward normal to $\partial\Omega$. It should be noted that N exists almost everywhere on the boundary by smoothness, and so boundary integrals involving N make sense. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution to the nonlinear elliptic problem

$$\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = f(u) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega$$

where f is continuous. Define $F(s) := \int_0^s f(\sigma)\ d\sigma$. In what follows the summation convention is observed on latin indices, but greek indices are not summed. Both run from 1 to n.

(a) For fixed α,

$$\int_\Omega x_\alpha \frac{\partial^2 u}{\partial x_\alpha \partial x_i} \frac{\partial u}{\partial x_i}\ dx = \int_\Omega x_\alpha \left(\frac{1}{2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} \right)_{x_\alpha} dx \tag{1}$$

$$= \frac{1}{2} \int_{\partial\Omega} x_\alpha \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} N_\alpha\ dS - \frac{1}{2} \int_\Omega (x_\alpha)_{x_\alpha} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i}\ dx \tag{2}$$

$$= \frac{1}{2} \int_{\partial\Omega} x_\alpha \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} N_\alpha\ dS - \frac{1}{2} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i}\ dx \tag{3}$$

where equality (2) is by Green's Theorem. Summing over α and noting that the second integrand does not depend on α yields the result

$$\int_\Omega x_j \frac{\partial^2 u}{\partial x_j \partial x_i} \frac{\partial u}{\partial x_i}\ dx = -\frac{n}{2} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i}\ dx + \frac{1}{2} \int_{\partial\Omega} (x \cdot N) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i}\ dS$$

(b) Let

$$F(s) = \int_0^s f(\sigma)\ d\sigma$$

So,

$$F'(s) = f(s)$$

For fixed α,

$$\int_\Omega F(u)\ dx = \int_\Omega (x_\alpha)_{x_\alpha} F(u)\ dx \tag{4}$$

$$= \int_{\partial\Omega} x_\alpha F(u) N_\alpha\ dS - \int_\Omega x_\alpha (F(u))_{x_\alpha} dx \tag{5}$$

$$= \int_{\partial\Omega} x_\alpha F(u) N_\alpha\ dS - \int_\Omega x_\alpha F'(u) \frac{\partial u}{\partial x_\alpha}\ dx \tag{6}$$

$$= \int_{\partial\Omega} x_\alpha F(u) N_\alpha\ dS - \int_\Omega x_\alpha \frac{\partial u}{\partial x_\alpha} f(u)\ dx \tag{7}$$

where equality (5) is by Green's Theorem, (6) is by the chain rule, and (7) is by our earlier computation. Summing over α, noting that the first integrand does not depend on α and rearranging yields the result

$$\int_\Omega f(u) x_i \frac{\partial u}{\partial x_i}\ dx = \int_{\partial\Omega} F(u) x_i N_i\ dS - n \int_\Omega F(u)\ dx$$

(c) Since $u = 0$ on $\partial\Omega$, $F(u) = F(0) = 0$ on $\partial\Omega$. Also, because Ω is open in $\mathbb{R}^n$, $\partial\Omega$ is an $(n-1)$–dimensional piecewise smooth hypersurface in $\mathbb{R}^n$. Thus the tangent space at a non-cusp point of $\partial\Omega$ is dimension $n-1$. Hence at a non-cusp point $x \in \partial\Omega$ there exists a basis for $\mathbb{R}^n$ which decomposes $\mathbb{R}^n$ as the orthogonal direct sum $\mathbb{R}^n = T_x(\partial\Omega) \oplus N$, where $T_x(\partial\Omega)$ is the tangent space to $\partial\Omega$ at x. Now, since $u = 0$ on $\partial\Omega$, the derivative of u in any direction tangent to $\partial\Omega$ is 0. It thus follows from the direct-sum decomposition above that $\nabla u = aN$ for some scalar a. In fact, $\nabla u = \frac{\partial u}{\partial N} N$, and so for any i,

$$\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial N} N_i$$

For fixed α,

$$\int_\Omega x_i \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_\alpha \partial x_\alpha}\ dx = \int_\Omega x_i \frac{\partial u}{\partial x_i} \left(\frac{\partial u}{\partial x_\alpha} \right)_{x_\alpha} dx \tag{8}$$

$$= \int_{\partial\Omega} x_i \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_\alpha} N_\alpha\ dS - \int_\Omega \frac{\partial u}{\partial x_\alpha} \left(x_i \frac{\partial u}{\partial x_i} \right)_{x_\alpha} dx \tag{9}$$

$$= \int_{\partial\Omega} x_i \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_\alpha} N_\alpha\ dS - \int_\Omega \left(x_i \frac{\partial u}{\partial x_\alpha} \frac{\partial^2 u}{\partial x_i \partial x_\alpha} + \frac{\partial u}{\partial x_\alpha} \frac{\partial u}{\partial x_\alpha} \right) dx \tag{10}$$

where equality (10) is by Green's Theorem and (11) is by the chain rule. Summing over α yields the result

$$\int_\Omega x_i \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_j \partial x_j}\ dx = \int_{\partial\Omega} x_i \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} N_j\ dS - \int_\Omega \left(x_i \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j}\right)\ dx$$

Now,

$$n\int_\Omega F(u)\ dx = \int_{\partial\Omega} F(u) x_i N_i\ dS - \int_\Omega f(u) x_i \frac{\partial u}{\partial x_i}\ dx \tag{11}$$

$$= -\int_\Omega x_i \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_j \partial x_j}\ dx \tag{12}$$

where (13) is by (b) and (14) is by $u = 0$ on $\partial\Omega$ and the PDE. From our above calculation we obtain

$$= -\int_{\partial\Omega} x_i \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} N_j\ dS + \int_\Omega x_i \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}\ dx + \int_\Omega \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j}\ dx \tag{13}$$

$$= -\int_{\partial\Omega} x_i \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} N_j\ dS + \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}\ dx - \frac{n}{2}\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}\ dx + \frac{1}{2}\int_{\partial\Omega} (x\cdot N)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}\ dS \tag{14}$$

$$= \frac{2-n}{2}\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}\ dx - \int_{\partial\Omega} \left(\frac{\partial u}{\partial N}\right) N_i \left(\frac{\partial u}{\partial N}\right) N_j N_j\ dS + \frac{1}{2}\int_{\partial\Omega} x_i N_i \left(\frac{\partial u}{\partial N}\right)^2 N_j N_j\ dS \tag{15}$$

$$= \frac{2-n}{2}\int_\Omega \|\nabla u\|^2\ dx - \frac{1}{2}\int_{\partial\Omega} \left(\frac{\partial u}{\partial N}\right)^2 (x\cdot N)\ dS \tag{16}$$

where (15) is by (12), (16) is by (a), and (17) is by (8). Rearranging yields the result

$$\frac{n-2}{2}\int_\Omega \|\nabla u\|^2\ dx + \frac{1}{2}\int_{\partial\Omega} \left(\frac{\partial u}{\partial N}\right)^2 (x\cdot N)\ dS + n\int_\Omega F(u)\ dx = 0$$

3. Consider the conservation law with 2 components (1 space variable) $\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}$, or equivalently $\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = \mathbf{0}$. Let $\lambda_1(\mathbf{u})$ and $\lambda_2(\mathbf{u})$ be the eigenvalues of $\mathbf{A}(\mathbf{u})$ with respective eigenvectors $\mathbf{r}_1$ and $\mathbf{r}_2$ and let $w_1(\mathbf{u})$ and $w_2(\mathbf{u})$ be corresponding Riemann invariants. Let $\mathbf{v}(x,t) := (v^1(x,t), v^2(x,t)) = \mathbf{w}(\mathbf{u}(x,t))$. Show that

$$v_t^1 + \lambda_2(\mathbf{u}) v_x^1 = 0$$

$$v_t^2 + \lambda_1(\mathbf{u}) v_x^2 = 0$$

4. Consider the conservation law with 2 components (1 space variable) $\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}$, or equivalently $\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = \mathbf{0}$. Let $\lambda_1(\mathbf{u})$ and $\lambda_2(\mathbf{u})$ be the eigenvalues of $\mathbf{A}(\mathbf{u})$ with respective eigenvectors $\mathbf{r}_1$ and $\mathbf{r}_2$ and let $w_1(\mathbf{u})$ and $w_2(\mathbf{u})$ be corresponding Riemann invariants. Let $\mathbf{v}(x,t) := (v^1(x,t), v^2(x,t)) = \mathbf{w}(\mathbf{u}(x,t))$. Show that

$$v_t^1 + \lambda_2(\mathbf{u}) v_x^1 = 0$$

$$v_t^2 + \lambda_1(\mathbf{u}) v_x^2 = 0$$

5. One-dimensional isentropic flow of a gas is modeled by the system

$$\rho_t + v\rho_x + \rho v_x = 0$$

$$v_t + v v_x + c^2 \rho_x/\rho = 0,$$

where c, which is positive, is the speed of sound and is a function of the density ρ, $c^2 = K\rho^{\gamma-1}$, $\gamma > 1$ is the ratio of the heat capacities (v is the velocity of the flow).

   (a) Write this system as a 2-dimensional conservation law $\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}$ for $\mathbf{u} = (\rho, v)^T$.

   (b) Write the system as $\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = \mathbf{0}$ and find the eigenvalues and eigenvectors of $\mathbf{A}(\mathbf{u})$).

(c) Introduce the function $L(\rho) = \int_0^\rho (c(\sigma)/\sigma)\, d\sigma$ and find Riemann invariants $q(\mathbf{u})$ corresponding to the eigenvalue λ_1 and $s(\mathbf{u})$ corresponding to the eigenvalue λ_2. Pick the labeling so that $\lambda_1 > \lambda_2$.

(d) Use problem 3 to find the system of equations for q and s.

(e) Show that the system can be "inverted" to obtain:

$$x_q = (v - c)t_q$$

$$x_s = (v + c)t_s$$

Hint:

$$\begin{pmatrix} \frac{\partial x}{\partial q} & \frac{\partial x}{\partial s} \\ \frac{\partial t}{\partial q} & \frac{\partial t}{\partial s} \end{pmatrix} = \begin{pmatrix} \frac{\partial q}{\partial x} & \frac{\partial q}{\partial t} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial t} \end{pmatrix}^{-1} = \frac{1}{D} \begin{pmatrix} \frac{\partial s}{\partial t} & -\frac{\partial q}{\partial t} \\ -\frac{\partial s}{\partial x} & \frac{\partial q}{\partial x} \end{pmatrix}, \quad D = \frac{\partial(q,s)}{\partial(x,t)}.$$

(f) Show that $v + c$ and $v - c$ can be written as functions of q and s and that the last system is therefore a fully linear system.

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- $$\begin{pmatrix} \rho \\ v \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \frac{1}{2}v^2 + \frac{K\rho^{\gamma-1}}{\gamma-1} \end{pmatrix}_x = 0$$

- $$\begin{pmatrix} \rho \\ v \end{pmatrix}_t + \begin{pmatrix} v & \rho \\ c^2/\rho & v \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_x = 0$$

- The characteristic equation for the above matrix is $\lambda^2 - 2v\lambda + v^2 - c^2 = 0$. This yields $\lambda_1 = v + c$ and $\lambda_2 = v - c$. The corresponding eigenvectors are $(\rho, c)^T$ and $(\rho, -c)^T$. We want $(q_\rho, q_v)^T \perp (\rho, c)$. This is easier to figure out if we separate the v and ρ in the eigenvector; that is to say, use the eigenvector $(1, c/\rho)$ instead. We then see that $q := v - L(\rho)$ works. Similarly, for λ_2 we get the Riemann invariant $s := v + L(\rho)$.

- From problem 3 we immediately get

$$q_t + (v - c)q_x = 0,$$
$$s_t + (v + c)s_x = 0.$$

- Since

$$\begin{pmatrix} D\frac{\partial x}{\partial q} & D\frac{\partial x}{\partial s} \\ D\frac{\partial t}{\partial q} & D\frac{\partial t}{\partial s} \end{pmatrix} = \begin{pmatrix} \frac{\partial s}{\partial t} & -\frac{\partial q}{\partial t} \\ -\frac{\partial s}{\partial x} & \frac{\partial q}{\partial x} \end{pmatrix},$$

we can rewrite the system as

$$x_q = (v + c)t_q,$$
$$x_s = (v - c)t_s.$$

- Using the fact that $q = v - 2c/(\gamma - 1)$ and $s = v + 2c/(\gamma - 1)$ we get $v = (q + s)/2$ and $c = (\gamma - 1)(s - q)/4$. A little algebra then yields

$$v + c = \frac{(3 - \gamma)q + (1 + \gamma)s}{4},$$

$$v + c = \frac{(3 - \gamma)s + (1 + \gamma)q}{4}.$$