MAT 576
Fall 2008
Homework set 4

1. In class we looked at the wave equation $u_{xx} - u_{yy} = 0$ on a bounded domain $\Omega \subset \mathbb{R}^2$ that is partitioned into two (open) subdomains Ω_1 to the left of a smooth curve $\mathcal{C} : x = s(t)$ and Ω_2 to the right. Letting $[[w]] := w|_{\Omega_2} - w|_{\Omega_1}$ we saw that a function u that satisfies $u_{xx} - u_{yy} = 0$ in $\Omega_1 \cup \Omega_2$ is a weak solution only if for all $\phi \in C_0^\infty(\Omega)$ we have

$$\int_{\mathcal{C}} ([[u_x]]\, N_1\phi - [[u_y]]\, N_2\phi)\, d\ell + \int_{\mathcal{C}} ([[u]]\, N_1\phi_x - [[u]]\, N_2\phi_y)\, d\ell = 0,$$

where $\langle N_1, N_2 \rangle$ is the (continuously varying) unit normal to $\mathcal{C}$. Show that the function $H(x-y)$ satisfies this condition.

2. This is an exercise in using Green's theorem (also known as Gauss's theorem or Stokes's theorem, or Ostrogradskii's theorem or by various combinations of these names):

$$\int_\Omega u_{x_i} v\, dx = -\int_\Omega u v_{x_i}\, dx + \int_{\partial\Omega} uvN_i\, dS,$$

where $N := \langle N_1, \cdots, N_n \rangle$ is the unit outward normal to $\partial\Omega$ and Ω a bounded domain in $\mathbb{R}^n$, $n > 1$, with piecewise smooth boundary. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and suppose that u is a solution to the nonlinear elliptic problem

$$\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Suppose that f is continuous and define $F(s) = \int_0^s f(\sigma)\, d\sigma$. We employ the summation convention.

   (a) Show that
   $$\int_\Omega x_j \frac{\partial^2 u}{\partial x_j \partial x_i} \frac{\partial u}{\partial x_i}\, dx = -\frac{n}{2} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i}\, dx + \frac{1}{2} \int_{\partial\Omega} (N \cdot x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i}\, dS.$$

   (b) Show that
   $$\int_\Omega f(u) x_i \frac{\partial u}{\partial x_i}\, dx = \int_{\partial\Omega} F(u) x_i N_i\, dS - n \int_\Omega F(u)\, dx.$$

   (c) Show that
   $$\frac{n-2}{2} \int_\Omega \|\nabla u|^2\, dx + \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial N}\right)^2 (x \cdot N)\, dS + n \int_\Omega F(u)\, dx\, dx = 0.$$

   Here $\partial u / \partial N = \nabla u \cdot N$ is the outward normal derivative. Hint: use the fact that $u = 0$ on the boundary - at least twice!

3. Consider the conservation law with 2 components (1 space variable) $\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}$, or equivalently $\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = \mathbf{0}$. Let $\lambda_1(\mathbf{u})$ and $\lambda_2(\mathbf{u})$ be the eigenvalues of $\mathbf{A}(\mathbf{u})$ with respective eigenvectors $\mathbf{r}_1$ and $\mathbf{r}_2$ and let $w_1(\mathbf{u})$ and $w_2(\mathbf{u})$ be corresponding Riemann invariants. Let $\mathbf{v}(x,t) := (v^1(x,t), v^2(x,t)) = \mathbf{w}(\mathbf{u}(x,t))$. Show that

$$v_t^1 + \lambda_2(\mathbf{u}) v_x^1 = 0$$

$$v_t^2 + \lambda_1(\mathbf{u}) v_x^2 = 0$$

4. One-dimensional isentropic flow of a gas is modeled by the system

$$\rho_t + v\rho_x + \rho v_x = 0$$

$$v_t + vv_x + c^2\rho_x/\rho = 0,$$

where c, which is positive, is the speed of sound and is a function of the density ρ, $c^2 = K\rho^{\gamma-1}$, $\gamma > 1$ is the ratio of the heat capacities (v is the velocity of the flow).

(a) Write this system as a 2-dimensional conservation law $\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}$ for $\mathbf{u} = (\rho, v)^T$.

(b) Write the system as $\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = \mathbf{0}$ and find the eigenvalues and eigenvectors of $\mathbf{A}(\mathbf{u})$).

(c) Introduce the function $L(\rho) = \int_0^\rho (c(\sigma)/\sigma)\, d\sigma$ and find Riemann invariants $q(\mathbf{u})$ corresponding to the eigenvalue λ_1 and $s(\mathbf{u})$ corresponding to the eigenvalue λ_2. Pick the labeling so that $\lambda_1 > \lambda_2$.

(d) Use problem 3 to find the system of equations for q and s.

(e) Show that the system can be "inverted" to obtain:

$$x_q = (v - c)t_q$$

$$x_s = (v + c)t_s$$

Hint:

$$\begin{pmatrix} \frac{\partial x}{\partial q} & \frac{\partial x}{\partial s} \\ \frac{\partial t}{\partial q} & \frac{\partial t}{\partial s} \end{pmatrix} = \begin{pmatrix} \frac{\partial q}{\partial x} & \frac{\partial q}{\partial t} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial t} \end{pmatrix}^{-1} = \frac{1}{D} \begin{pmatrix} \frac{\partial s}{\partial t} & -\frac{\partial q}{\partial t} \\ -\frac{\partial s}{\partial x} & \frac{\partial q}{\partial x} \end{pmatrix}, \quad D = \frac{\partial(q, s)}{\partial(x, t)}.$$

(f) Show that $v + c$ and $v - c$ can be written as functions of q and s and that the last system is therefore a fully linear system.