Classification of higher order equations - Characteristics.

Classification of PDEs is founded on the notion of *characteristic*. We already talked about classification of first order systems and (scalar) second order equations. In this little note we will look at semilinear higher order equations on a open domain $\mathcal{D} \subset \mathbb{R}^n$:

$$\sum_{|\alpha|=m} a_\alpha(x) D_x^\alpha u = F, \tag{1}$$

where F is a function of x, u, and derivatives of u of order less than m and D_x indicates derivatives with respect to x. Let $\mathcal{S}$ be a $(n-1)$-dimensional smooth surface in $\mathcal{D}$ and P a point on this surface. We need to define what it means for the surface to be noncharacteristic at P. To do this we first change coordinates $\xi_k = \psi(x_1, \cdots, x_n)$ in such a way that $\xi_n = 0$ is an equation for $\mathcal{S}$ near P, and the Jacobian matrix $(\partial\xi_i/\partial x_j)$is nonsingular. The inverse transformation is given by $x_j = X_j(\xi_1, \cdots, \xi_n)$. This means that for fixed $\xi_1, \cdots, \xi_{n-1}$ the equations $x = X(\xi_1, \cdots, \xi_{n-1}, \xi_n)$, $-\epsilon < \xi_n < \epsilon$ are parametric equations for curves that intersect $\mathcal{S}$ transversally. Now we suppose that we are given the values of u, $\partial u/\partial \xi_n$, ... ,$\partial^{m-1}u/\partial\xi_n^{m-1}$ on the surface $\mathcal{S}$. We say that the surface is noncharacteristic at P if this data, together with the PDE, allows us to find $\partial^m u/\partial\xi_n^m$ at P. The chain rule gives us $\partial u/\partial x_k = \sum_i u_{x_i}\partial\xi_i/\partial x_k$ and for second order derivatives we have

$$\frac{\partial^2 u}{\partial x_k \partial x_l} = \sum_{i,j=1}^{n} u_{\xi_i\xi_j}\frac{\partial \xi_i}{\partial x_k}\frac{\partial \xi_j}{\partial x_l} + \sum_{i,j=1}^{n} u_{x_i}\frac{\partial^2 \xi_i}{\partial x_k \partial x_l}$$

which can be written using the summation convention as

$$\frac{\partial^2 u}{\partial x_k \partial x_l} = u_{\xi_i\xi_j}\frac{\partial \xi_i}{\partial x_k}\frac{\partial \xi_j}{\partial x_l} + \text{ terms with lower order derivatives of } u.$$

In general

$$\frac{\partial^N u}{\partial x_{k_1}\partial x_{k_2}\cdots\partial x_{k_N}} = u_{\xi_{\ell_1}\xi_{\ell_2}\cdots\xi_{\ell_N}}\frac{\partial \xi_{\ell_1}}{\partial x_{k_1}}\frac{\partial \xi_{\ell_2}}{\partial x_{k_2}}\cdots\frac{\partial \xi_{\ell_N}}{\partial x_{k_N}} + \text{ terms with lower order derivatives of } u.$$

We are interested in the derivatives of order m, and particularly the m^{th} order derivative with respect to ξ_n. Note that if $|\alpha| = m$ then

$$D_x^\alpha u = \frac{\partial^m u}{\partial \xi_n^m}\left(\nabla\xi_n\right)^\beta + \cdots$$

where the terms that are not explicitly written down only contain derivatives whose orders with respect to ξ_n are less than m and whose values are therefore known on $\mathcal{S}$. Substituting these into equation (1) we have

$$\left(\sum_{|\alpha|=m} a_\alpha(x)\left(\nabla\xi_n\right)^\alpha\right)\frac{\partial^m u}{\partial\xi_n^m} = F + \cdots, \tag{2}$$

where everything on the right hand side is known on $\mathcal{S}$. Therefore we can solve for $\partial^m u/\partial\xi_n^m$ provided

$$\left(\sum_{|\alpha|=m} a_\alpha(x)\left(\nabla\xi_n\right)^\alpha\right) \neq 0.$$

Since $\nabla\xi_n$ is colinear with the unit normal $\mathbf{n}$ to the surface we see that $\mathcal{S}$ is noncharacteristic at a point P provided

$$\left(\sum_{|\alpha|=m} a_\alpha(P)\left(\mathbf{n}(P)\right)^\alpha\right) \neq 0.$$

In particular, equation (1) is elliptic at P (i.e. has no characteristic surfaces there) if

$$\left(\sum_{|\alpha|=m} a_\alpha(P)\mathbf{v}^\alpha\right) \neq 0 \quad \forall\, 0 \neq \mathbf{v} \in \mathbb{R}^n.$$

The generalization to higher order systems is natural:
$\mathbf{u}$ is a vector-valued function $(u_1(x), u_2(x), \cdots, u_M(x))^T$, equation (1) becomes a system $L\mathbf{u} = \mathbf{F}$ where for each α, a_α is an $M \times M$ matrix. The surface $\mathcal{S}$ is characteristic at P if for a nonzero unit normal $\mathbf{n}(P)$ to $\mathcal{S}$ at the point P we have

$$\det\left[\left(\sum_{|\alpha|=m} a_\alpha(P)\left(\mathbf{n}(P)\right)^\alpha\right)\right] \neq 0.$$

We note that *hyperbolicity* is usually restricted to first-order systems and second order scalar equations. For these cases we have seen the definitions of hyperbolicity in class. The term *parabolicity* is usually restricted to second order equations and systems of coupled second order parabolic equations such as *reaction-diffusion* systems: $\mathbf{u}_t = -L\mathbf{u} + \mathbf{F}$.