A Note on some inequalities.

A basic inequality is *Young's inequality*: If a, b are nonnegative numbers and $1 \le p < \infty$ then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

This inequality can be proven by elementary means (when $b \neq 0$ let $x = a^{p-1}/b$ and then find the minimum of the function $f(x) = x/p + (p-1)x^{-1/(p-1)}/p$). One consequence of Young's inequality is *Hölder's inequality*. For vectors this inequality says

$$\left| \sum_{j=1}^{n} x_j y_j \right| \le \|x\|_p \|y\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

To prove this we use Young's inequality

$$\frac{|x_j|}{\|x\|_p} \frac{|y_j|}{\|y\|_p} \le \frac{1}{p} \frac{|x_j|}{\sum |x_j|^p} + \frac{1}{q} \frac{|y_j|}{\sum |y_j|^q}.$$

When we sum over i we have Hölder's inequality for vectors. Similarly Hölder's inequalityfor functions reads

$$\left| \int_\Omega f(x) g(x)\, dx \right| \le \|f\|_p \|g\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Here

$$\|f\|_p = \sqrt[p]{\int_\Omega |f(x)|^p \, dx}.$$

To prove this inequality we do essentially the same thing:

$$\frac{|f(x)|}{\|f\|_p} \frac{|g(x)|}{\|g\|_q} \le \frac{1}{p} \left(\frac{|f(x)|}{\|f\|_p} \right)^p + \frac{1}{q} \left(\frac{|g(x)|}{\|g\|_q} \right)^q.$$

After integrating this inequality we get the desired result. Note that Hölder's inequality is true even if we set $p = \infty$ and $q = 1$.

We can apply Hölder's inequality twice and deduce that

$$\left| \int_\Omega f(x) g(x) h(x)\, dx \right| \le \|f\|_p \|g\|_q \|h\|_r$$

when $1 \le p, q, r < \infty$ and

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,$$

and so on.

Recall the definition of the convolution:

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y) g(y)\, dy.$$

We want to prove *Young's inequality for convolutions:*

$$\|f * g\|_s \leq \|f\|_p \, \|g\|_q, \text{ where } \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{s}.$$

This formula generalizes the well known formula $\|f * g\|_1 \leq \|f\|_1 \, \|g\|_1$ for L_1 norms. Note that

$$\int_{\mathbb{R}} h(x)(f * g)(x)\, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} h(x) f(x-y) g(y)\, dy\, dx,$$

We next define

$$\begin{aligned} \alpha(x,y) :=& \quad |h(x)|^{p/q'}/, |f(x-y)|^{p/q'} \\ \beta(x,y) :=& \quad |f(x-y)|^{p/r'}/, |g(y)|^{q/r'} \\ \gamma(x,y) :=& \quad |g(y)|^{q/p'}/, |h(x)|^{r/p'} \end{aligned}$$

so that after some algebra with the exponents:

$$\alpha(x,y)\beta(x,y)\gamma(x,y) = |h(x)f(x-y)g(y)|.$$

Now applying Hölder's inequality for three functions we have

$$\left| \int_{\mathbb{R}} h(x)(f * g)(x)\, dx \right| \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha(x,y)\beta(x,y)\gamma(x,y)\, dx\, dy \leq \|\alpha\|_{q'} \|\beta\|_{r'} \|\gamma\|_{p'} = \|f\|_p \|g\|_q \|h\|_r.$$

Letting $\theta(x)$ denote the phase of $(f * g)(x)$ we see that upon setting

$$h(x) = |(f * g)(x)|^{r'/r} \exp(-i\theta(x,y))$$

we obtain

$$\left| \int_{\mathbb{R}} |(f * g)(x)|^{r'}\, dx \right| \leq \|f\|_p \|g\|_q \|f * g\|_{r'}^{r'/r}.$$

and doing some algebra and setting $r' = s$:

$$\|f * g\|_s \leq \|f\|_p \|g\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{s}.$$

Application to sequences and difference equations

The above proofs are actually independent of the measure on the real line and hence are equally true if dx is replaced by $d\mu$ where μ is any Borel measure. In particular we can apply it to the case where μ is the "counting" measure: $\mu(E)$ = number of integers in E. Then

$$\|f\|_p = \sqrt[p]{\sum_{j=-\infty}^{\infty} |f(j)|^2}.$$

Of course we may use the notation $f_j = f(j)$ as we usually do for sequences. The convolution $h = f * g$ is then given by

$$h(k) = \sum_{j=-\infty}^{\infty} f(k-j)g(j) = \sum_{j=-\infty}^{\infty} f(j)g(k-j).$$

If

$$M := \|g\|_1 = \sum_{j=-\infty}^{\infty} |g(j)| < \infty$$

then $\|f * g\|_2 \leq M\|f\|_2$. In particular suppose that α is a number such that $|\alpha| < 1$ and let us define g as follows: $g(j) = 0$ for all $j < 0$ and $g(j) = \alpha^j)$ for all $j \geq 0$. Then

$$\sum_{j=-\infty}^{\infty} g(j) = \sum_{j=0}^{\infty} \exp(-\gamma j) = 1/[1-\alpha)]$$

and hence if

$$\|f\|^2 = \sum_{j=-\infty}^{\infty} |f(j)|^2 < \infty$$

and

$$h(k) = (f * g)(k) = \sum_{j=-\infty}^{k} f(j)g(k-j),$$

then

$$\|h\|_2 \leq \|f\|_2/[1-\alpha].$$

This result is useful in solving certain difference equations. For example, consider the difference equation

$$x_{j+1} = \alpha x_j + f_{j+1},$$

where α is a constant with $|\alpha| < 1$. If we multiply this equation by α^{-j-1} and set $y_j := \alpha^{-j} x_j$ we get

$$y_{j+1} = y_j + f_{j+1}\alpha^{-j-1},$$

and this has the solution

$$y_j = \sum_{k=-\infty}^{j} f_k \alpha^{-k},$$

provided this sum converges! Going back to x_j:

$$x_j = \sum_{k=-\infty}^{j} f_k \alpha^{j-k}.$$

Let us define $g_j = \alpha^j$ for $j \geq 0$ and $g_j = 0$ for $j < 0$. Then our solution x is precisely $f * g$ and therefore we have

$$\|x\|_2 \leq \frac{\|f\|_2}{1-\alpha}.$$