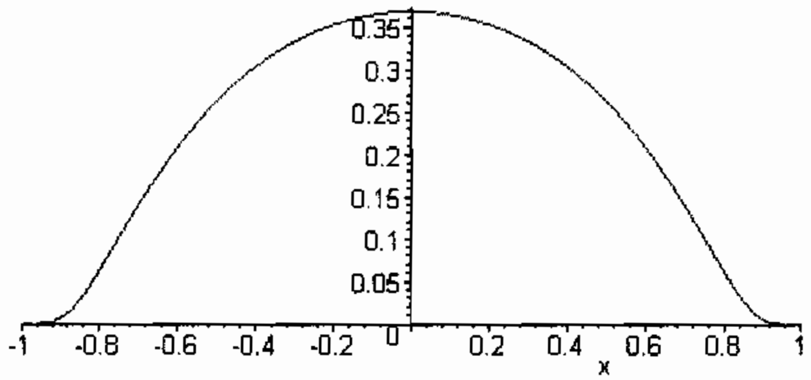
The Mollifier Theorem

Definition of the Mollifier

The function

$$T(x) = \begin{cases} K\exp\left(\frac{-1}{1-|x|^2}\right) & if \ \ |x| < 1 \\ 0 & if \ \ |x| \geq 1 \end{cases}, \qquad x \in R^n$$

where the constant K is chosen such that $\int_{R^n} T(x)dx = 1$, is a test function on R^n. Note that $T(x)$ vanishes, together with all its derivatives as $|x| \to 1^-$, so $T(x)$ is infinitely differentiable and has compact support. The graph of $T(x)$ is sketched in the following figure.



The Mollifier Function

.For $n = 1$ *and* $\varepsilon > 0$, let

$$S_\varepsilon(x) = \frac{1}{\varepsilon} T\left(\frac{x}{\varepsilon}\right) \qquad and \qquad P_\epsilon(x) = T\left(\frac{x}{\varepsilon}\right).$$

Then

$$S_\varepsilon(x) \geq 0 \qquad and \qquad P_\epsilon(x) \geq 0 \quad for\ all\ x$$

$$S_\varepsilon(x) = 0 \qquad and \qquad P_\epsilon(x) = 0 \quad for\ |x| > \varepsilon$$

$$\int_R S_\varepsilon(x)\,dx = 1 \qquad \forall \varepsilon > 0, \quad S_\varepsilon(0) \to +\infty \quad as \quad \varepsilon \to 0,$$

$$\int_R P_\varepsilon(x)\,dx \to 0 \qquad as \quad \varepsilon \to 0, \quad P_\varepsilon(0) = K/e \qquad \forall \varepsilon > 0,$$

Evidently, $S_\varepsilon(x)$ becomes thinner and higher as ε tends to zero but the area under the graph is constantly equal to one. On the other hand, $P_\epsilon(x)$ has constant height but grows thinner as ε tends to zero. These test functions can be used as the "seeds" from which an infinite variety of other test functions can be constructed by using a technique called regularization which we will now describe.

For $n \geq 1$ we have

$$S_\varepsilon(x) = \frac{1}{\varepsilon^n} T\left(\frac{x}{\varepsilon}\right) \qquad and \qquad P_\epsilon(x) = T\left(\frac{x}{\varepsilon}\right).$$

For U a bounded open set in R^n, and for $u \in L^1_{loc}(U)$, define for any $\varepsilon > 0$ and any $x \in U_\varepsilon = \{x \in U : dist(x, \partial U) > \varepsilon\}$,

$$J_\varepsilon u(x) = \int_{|x-y|\le\varepsilon} S_\varepsilon(x-y)\,u(y)\,dy \qquad (1.1a)$$

$$= \int_{|z|\le\varepsilon} S_\varepsilon(z)\,u(x-z)\,dz \qquad (1.1b)$$

$$= \int_{|z|\le 1} S_1(z)\,u(x-\varepsilon z)\,dz. \qquad (1.1c)$$

We refer to $J_\varepsilon u(x)$ as the ***mollified*** $u(x)$. This mollified function, $J_\varepsilon u(x)$, is a smoothed version of the original function, $u(x)$.

Properties of the Mollifier

Note first that $J_\varepsilon u(x)$ is infinitely differentiable; i.e., for any $\varepsilon > 0$ and any $x \in U_\varepsilon$, it is clear from (1.1a) that

$$\frac{J_\varepsilon u(x+\varepsilon\vec{e}_i) - J_\varepsilon u(x)}{\varepsilon} = \int_{|x-y|\le\varepsilon} \frac{[S_\varepsilon(x+\varepsilon\vec{e}_i - y) - S_\varepsilon(x-y)]}{\varepsilon}\,u(y)\,dy$$

i.e.,

$$\frac{J_\varepsilon u(x+\varepsilon\vec{e}_i) - J_\varepsilon u(x)}{\varepsilon} \to \int_{|x-y|\le\varepsilon} \partial_{x_i} S_\varepsilon(x-y)\,u(y)\,dy \qquad as\ \varepsilon \to 0.$$

Since $S_\varepsilon(x)$ is infinitely differentiable, it follows that $J_\varepsilon u(x)$ is infinitely differentiable on the open set U_ε.

It is evident from (1.1a) that for $1 \le p < \infty,\ \varepsilon > 0,\ and\ \ x \in U_\varepsilon$,

$$J_\varepsilon u(x) = \int_{|x-y|\le\varepsilon} S_\varepsilon(x-y)^{1-1/p} S_\varepsilon(x-y)^{1/p}\,u(y)\,dy.$$

Then, using Holder's inequality, we get

$$|J_\varepsilon u(x)|^p = \left(\int_{|x-y|\le\varepsilon} S_\varepsilon(x-y)\,dy\right)^{p-1} \int_{|x-y|\le\varepsilon} S_\varepsilon(x-y)|\,u(y)|^p\,dy$$

and since $\int_R S_\varepsilon(x)\,dx = 1 \qquad \forall\varepsilon > 0$,

$$\int_V |J_\varepsilon u(x)|^p dx \le \int_V \int_{|x-y|\le\varepsilon} S_\varepsilon(x-y)|\,u(y)|^p\,dy\,dx$$

$$= \int_W |\,u(y)|^p \int_{|x-y|\le\varepsilon} S_\varepsilon(x-y)dx\,dy = \int_W |\,u(y)|^p dy$$

for open sets $W = U_\varepsilon,\ and\ V = W_\varepsilon$. This result is just that assertion that

$$\|J_\varepsilon u\|_{L_p(V)} \le \|u\|_{L_p(W)} \quad for \quad V \subset\subset W \subset\subset U \qquad (1.2)$$

Next, use (1.1c) to write

$$J_\varepsilon u(x) - u(x) = \int_{|z|\le 1} S_1(z)\,[u(x-\varepsilon z) - u(z)]\,dz.$$

If the function $u = u(x)$ is, in fact, continuous on U, then this last result shows that

$$\max_{\bar V} |J_\varepsilon u(x) - u(x)| \le \max_{\bar V} |[u(x-\varepsilon z) - u(z)]| \to 0 \ as\ \varepsilon \to 0; \qquad (1.3)$$

i.e., $J_\varepsilon u(x)$ converges uniformly to $u(x)$ for $x \in \bar V$ when $u(x)$ is continuous on $.U$.

For $u \in L^p_{loc}(U),\ W = U_\varepsilon,$ and arbitrary $\delta > 0$, use the fact that the continuous functions are dense in $L_p(W)$ to choose $v \in C(W)$ such that

$$\|u - v\|_{L_p(W)} \le \delta.$$

Then for $V = W_\varepsilon$,

$$\|J_\varepsilon u - u\|_{L_p(V)} \le \|J_\varepsilon u - J_\varepsilon v\|_{L_p(V)} + \|J_\varepsilon v - v\|_{L_p(V)} + \|v - u\|_{L_p(V)}$$

$$\le \|u - v\|_{L_p(W)} + \|J_\varepsilon v - v\|_{L_p(V)} + \|v - u\|_{L_p(W)} \le 2\delta + \|J_\varepsilon v - v\|_{L_p(V)}$$

It follows now from (1.3) that for $u \in L^p_{loc}(U)$,

$$\forall V \subset\subset U, \qquad \|J_\varepsilon u - u\|_{L^p(V)} \to 0 \quad as\ \varepsilon \to 0 \qquad (1.4)$$

We can summarize these results in the following,

Theorem (Local Approximation) Suppose U is open and bounded in R^n, $1 \le p < \infty$, and for $\varepsilon > 0$, let U_ε denote the subset $\{x \in U : dist(x, \partial U) > \varepsilon\}$.

(a) For every $\epsilon > 0$, $u \in L^p_{loc}(U)$ implies $J_\epsilon u \in C^\infty(U_\epsilon)$

(b) (i) $u \in C(U)$ implies u_ε converges to u uniformly on compact subsets of U; i.e.,

$$\|J_\varepsilon u - u\|_{C(\bar V)} = \max_{\bar V} |J_\epsilon u(x) - u(x)| \to 0 \text{ for all } V \subset\subset U$$

(ii) u_ε converges to u in $L^p_{loc}(U)$; i.e., $u \in L^p_{loc}(U)$ implies that for all $V \subset\subset W \subset\subset U$,

$$\| J_\epsilon u \|_{L^p(V)} \le \| u \|_{L^p(W)} \qquad \text{and} \qquad \| J_\epsilon u - u \|_{L^p(V)} \to 0 \text{ as } \epsilon \to 0$$

(c) u_ε converges to u in $W^{k,p}_{loc}(U)$;

Result (c) follows from (b) by induction.

Corollary (Global Approximation) Suppose U has a smooth boundary, and $1 \le p < \infty$.

(a) For every $\epsilon > 0$, $u \in L^p(U)$ implies $J_\epsilon u \in C^\infty(U) \cap L^p(U)$.

(b) $u \in L^p(U)$ implies that

$$\| J_\epsilon u \|_{L^p(U)} \le \| u \|_{L^p(U)} \qquad \text{and} \qquad \| J_\epsilon u - u \|_{L^p(U)} \to 0 \text{ as } \epsilon \to 0$$

(c) $u \in W^{k,p}(U)$ implies that there exists functions $\{\phi_m\} \in C^\infty(U) \cap W^{k,p}(U)$ such that

$$\|\phi_m - u\|_{k,p} \to 0 \qquad as\ m \to \infty.$$

The proof of the corollary makes use of a partition of unity (see theorem 2 pg 251 in Evans).

Weak Equals Strong

For U a bounded open set in R^n, we define $v = v(x)$ to be the ***weak derivative*** of order α, of $u = u(x),\ x \in U$ *if*

$$\int_U u(x)\, \partial^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_U v(x)\, \phi(x)\, dx \text{ for all } \phi \in C_c^\infty(U)$$

Similarly, we define $v = v(x)$ to be the ***strong L_p –derivative*** of order α, of $u = u(x),\ x \in U$ *if*

for any $V \subset\subset U$, there exists a sequence $\{\phi_n\} \in C_c^\infty(U)$ such that

$$\int_V |\phi_n - u|^p dx \to 0 \quad and \quad \int_V |\partial^\alpha \phi_n - v|^p dx \to 0, \quad as \quad n \to \infty.$$

Using mollifiers, we can show that these two notions are equivalent.

Suppose first that $v = v(x)$ is the weak derivative of order α, of $u = u(x)$. Then, since $S_\varepsilon \in C_c^\infty(U)$,

$$\partial^\alpha J_\varepsilon u(x) = \int_{|x-y|\le\varepsilon} \partial_x^\alpha S_\varepsilon(x-y)\, u(y)\, dy = (-1)^{|\alpha|} \int_{|x-y|\le\varepsilon} \partial_y^\alpha S_\varepsilon(x-y)\, u(y)\, dy$$

$$= \int_{|x-y|\le\varepsilon} S_\varepsilon(x-y)\, v(y)\, dy = J_\varepsilon v(x) \qquad \text{(by definition of weak derivative)}$$

Now apply (1.4) to write

$$\int_V |J_\varepsilon u - u|^p dx \to 0 \quad and \quad \int_V |\partial^\alpha J_\varepsilon u - v|^p dx = \int_V |J_\varepsilon v - v|^p dx \to 0, \quad as \quad n \to \infty.$$

Thus every weak derivative is a strong L_p –derivative.

Conversely, suppose $v = v(x)$ is the strong L_p –derivative of order α, of $u = u(x)$ with

$$\int_V |\phi_n - u|^p dx \to 0 \quad and \quad \int_V |\partial^\alpha \phi_n - v|^p dx \to 0, \quad as \quad n \to \infty,$$

for arbitrary $V \subset\subset U,\ and\ \{\phi_n\} \in C_c^\infty(U)$. Then for any $\psi \in C_c^\infty(U)$,

$$\int_V (u - \phi_n)\, \partial^\alpha \psi\, dx = \int_V u\, \partial^\alpha \psi\, dx - \int_V \phi_n\, \partial^\alpha \psi\, dx$$

$$= \int_V u\, \partial^\alpha \psi\, dx - (-1)^{|\alpha|} \int_V \partial^\alpha \phi_n\, \psi\, dx$$

$$= \int_V u\, \partial^\alpha \psi\, dx - (-1)^{|\alpha|} \int_V v\, \psi\, dx + (-1)^{|\alpha|} \int_V (v - \partial^\alpha \phi_n)\, \psi\, dx$$

Then it follows that

$$\left| \int_V u\, \partial^\alpha \psi\, dx - (-1)^{|\alpha|} \int_V v\, \psi\, dx \right| \le C_1 \int_V |\phi_n - u|^p dx + C_2 \int_V |\partial^\alpha \phi_n - v|^p dx$$

which implies that every strong L_p –derivative is a weak derivative.

Weyl's Lemma

Weyl's lemma is a famous result that asserts that for U a bounded open set in R^n, if $u = u(x)$ is harmonic in U, (i.e., $u \in C^2(U)$ *and* $\nabla^2 u(x) = 0,\ x \in U$) then $u(x)$ is infinitely differentiable in U.

To see why this result is true, recall that every harmonic function has the mean value property. That is,

$$\forall x \in U_\varepsilon,\ r < \varepsilon, \qquad u(x) = \int_{\partial B_r(x)} u(y)\, d\hat{S}(y) = \frac{1}{nr^{n-1}A_n} \int_{\partial B_r(x)} u(y)\, dS(y).$$

Then

$$J_\varepsilon u(x) = \int_{|x-y|\le\varepsilon} S_\varepsilon(x-y)\, u(y)\, dy = \frac{1}{\varepsilon^n} \int_{|x-y|\le\varepsilon} T\Big(\frac{x-y}{\varepsilon}\Big)\, u(y)\, dy$$

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon T\Big(\frac{r}{\varepsilon}\Big) \int_{\partial B_r(x)} u(y)\, dS(y)\ dr = u(x) \int_0^\varepsilon \frac{nA_n}{\varepsilon^n} T\Big(\frac{r}{\varepsilon}\Big)\, r^{n-1} dr$$

$$= u(x) \int_{B_\varepsilon(0)} S_\varepsilon(y)\, dy = u(x).$$

But this says that $\forall\varepsilon > 0,\ \ \forall x \in U_\varepsilon,\ J_\varepsilon u(x) = u(x)$. Since $J_\varepsilon u(x)$ is infinitely differentiable on U, it follows that $u(x)$ is infinitely differentiable on U although u need not even be continuous on the closure, $\bar{U}$.