Theory of Distributions

Recall: X = linear space + topology

(∴ continuity makes sense) X' = all continuous linear functionals

Sometimes X' can be represented by a space Y acting on X

∴ $\forall\, l \in X'\ \exists\ y_l \in Y$ s.t. $l(x) = \langle y_l, x\rangle$
where ~~y_l~~ $\langle\, , \rangle$ is a bilinear functional.

<u>Example</u> $\mathbb{R}^n$ $\quad l : \mathbb{R}^n \to \mathbb{R}$ linear f^{cnal}
$\Rightarrow \exists\ y_l \in \mathbb{R}^n$ s/t $\ l(x) = y_l^T x \quad \forall x$

$$\langle y_l, x\rangle = y_l^T x.$$

<u>Example</u> H Hilbert space $\forall\, l \in H'\ \exists$
$y_l \in H$ s.t. $\ l(x) = \langle x, y_l\rangle$

<u>Example</u> Banach space $L_p(\Omega)$ $\quad 1 \le p < \infty$
<u>Thm</u> $\forall\, l \in L_p(\Omega)'\ \exists\ g_l \in L_q(\Omega)$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$
s/t

$$l(f) = \langle g_l, f\rangle = \int_\Omega f(x)\,\overline{g_l(x)}\,dx$$

Also $\|l\| = \|g_l\|_{L_q}$

<u>Example</u> K cpct in $\mathbb{R}^n$ $\quad C(K)$ is B-space
$\forall\ l \in C(K)'\ \exists$ complex regular Borel measure μ_l
s/t $\ l(f) = \int_K f(x)\,d\mu_l$

Linear space of test functions
(Until further notice all f^{ons} are real valued)

$\mathcal{D} = \mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$

We may restrict to test functions on $\Omega \subset \mathbb{R}^n$

$\mathcal{D}(\Omega) = C_0^\infty(\Omega)$

First question. Is $C_0^\infty(\Omega)$ nonempty?

Example ~~$T(x) =$~~ $y = T(x) = \begin{cases} e^{+\frac{1}{x^2-1}} & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$

$T(x) = \exp((\tau(x))^{-1}) \qquad \tau(x) = x^2 - 1$

Easy to show by induction

$$T^{(m)}(x) = \exp(\tau(x)^{-1}) \sum_{j=0}^{N_m} \frac{p_j(x)}{(1-x^2)^j} \qquad |x|<1$$

where p_j is a polynomial

Since $\lim_{\alpha \uparrow \infty} e^{-\alpha} \alpha^k = 0 \quad k = 0, 1, 2, \ldots$ we see

then $\lim_{|x| \to 1-} T^{(m)}(x) = 0 \quad \therefore T^{(m)}$ contin. $\forall m$

$\therefore T \in C_0^\infty(\mathbb{R})$

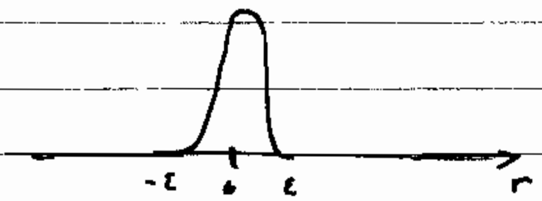
$\therefore T(x)\, T(y) \in C_0^\infty(\mathbb{R}^2)$ etc.

Mollifiers

$$T(x) = \begin{cases} K e^{\frac{1}{|x|^2-1}} & |x|<1 \\ 0 & |x| \geq 0 \end{cases}$$

K s.t. $\int_{\mathbb{R}^n} T(x)\,dx = 1$

$$S_\varepsilon(x) = \frac{1}{\varepsilon^n} T\left(\frac{x}{\varepsilon}\right)$$

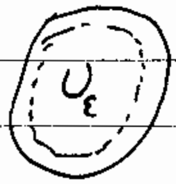


$$\int_{\mathbb{R}^n} S_\varepsilon(x)\,dy = 1$$

$$(J_\varepsilon f)(x) = \int S_\varepsilon(x-y)\, f(y)\,dy \quad \in$$

$J_\varepsilon f \in C_0^\infty(\mathbb{R}^n)$ if $f \in C(\mathbb{R}^n)$

f cont. on cpct set $K \Rightarrow J_\varepsilon f \to f$ unif. on K



Thm $U \subset \mathbb{R}^n$ with ∂U smooth. 1Space

$U_\varepsilon \stackrel{\text{def}}{=} \{x \in U \mid \text{dist}(x, \partial U) > \varepsilon\}$

a) $\forall \varepsilon > 0$ $u \in L^p(U) \Rightarrow J_\varepsilon u \in C^\infty(U) \cap L^p(U)$ and $\|J_\varepsilon u - u\|_{L_p} \to 0$ as $\varepsilon \to 0$

b) $u \in C(U) \Rightarrow \|u_\varepsilon - u\|_{C(K)} \to 0$ as $\varepsilon \to 0$ if $K \subset U$ K cpct.

Multi-index

$$\varphi : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$$

$\mathbb{Z}_+ = \{0, 1, 2, \ldots \}$

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{Z}_+^m$

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_m}}{\partial x_m^{\alpha_m}}$$

$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$

$|\alpha|$ = order of the differential operation D^α

Note If $f \in C^\infty(\Omega)$ $\varphi \in C_0^\infty(\Omega)$
$\Rightarrow f\varphi \in C_0^\infty(\Omega)$

Linear diff. operator $\sum_{|\alpha| \le p} a_\alpha(x) D^\alpha$

Formal adjoint $L^* f = \sum_{|\alpha| \le p} (-1)^{|\alpha|} D^\alpha (a_\alpha f)$

Note $f \in C^p(\Omega)$ $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega (Lf)\varphi \, dx = \int_\Omega f (L^* \varphi) \, dx$$

L^* = formal adjoint.

Examples (i) $L = -\frac{d}{dx} p(x) \frac{d}{dx} + q(x)$ $L^* = L$

(ii) $L = \frac{d}{dx}$ $L^* = -\frac{d}{dx}$

The above differs from strict adjoints.

$L : \mathcal{D}_B \subset L_2(\Omega) \to L_2(\Omega)$ (order p)

$$\mathcal{D}_B = \{ f \in C^p(\bar{\Omega}) \mid Bf = 0 \text{ certain B.C. } \}$$

$\exists$ adjoint boundary conditions B^* s.t. if we define L^* on $\mathcal{D}_{B^*}$ then

$$\langle Lf, g \rangle = \langle f, L^* g \rangle \quad \text{whenever}$$

$f \in \mathcal{D}_B$, $g \in \mathcal{D}_{B^*}$. The strict adjoint of L is the closure of $L^* : \mathcal{D}_{B^*} \to L_2(\Omega)$.

This is covered in an advanced PDE course.

<u>Definition</u> $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{C}$, or $\mathbb{R}^k$, or $\mathbb{C}^k$)
<u>supp(f)</u> = closure of $\{x \mid f(x) \neq 0\}$

Convergence

Recall compact set in $\mathbb{R}^n$ is a closed & bounded set

$\{\varphi_k\}$ sequence in $C_0^\infty(\Omega)$ is said to converge to $\varphi \in C_0^\infty(\Omega)$ if

(i) $\exists$ cpct. set K s.t. $\operatorname{supp}(\varphi_k) \subset K$ $\forall k$
and $\operatorname{supp}(\varphi) \subset K$
(ii) $D^\alpha \varphi_m \to D^\alpha \varphi$ uniformly on K.

<u>Definition</u> A null sequence is a sequence of test functions that converges to zero in the above sense

This type of convergence defines the topology on the test functions

<u>Definition</u> $\mathcal{D}(\Omega)'$ denotes the ~~vector~~ linear space of all continuous linear functionals on $\mathcal{D}(\Omega)$. Elements of $\mathcal{D}(\Omega)'$ are called "distributions on Ω".

<u>Note</u> ~~[illegible]~~ A linear f^nal ℓ on $\mathcal{D}(\Omega)$ is continuous iff $\ell(\varphi_m) \to 0$ $\forall$ null sequence $\{\varphi_m\}$.

Operators on distributions

i) Translation $\quad \ell \in \mathcal{D}(\Omega)'$

$$\ell_a(\varphi) = \ell(\varphi_{-a}) \qquad \varphi_{-a}(x) = \varphi(x+a)$$

$$\langle f_a, \varphi\rangle = \int f(x)\varphi(x+a)\,dx = \int f(x-a)\varphi(x)\,dx$$

$$\therefore f_a(\cdot) = f(\cdot - a)$$

ii) Differentiation $\quad \left(\frac{\partial}{\partial x_k}\ell\right)(\varphi) = -\ell\left(\frac{\partial \varphi}{\partial x_k}\right)$

If $f \in C^1(\Omega)$

$$\frac{\partial \ell}{\partial x_k}(\varphi) = \left\langle \frac{\partial f}{\partial x_k}, \varphi\right\rangle = \int_\Omega \frac{\partial f}{\partial x_k}\varphi\,dx = -\int_\Omega f\frac{\partial \varphi}{\partial x_k}\,dx$$

$$= -\left\langle \ell, \frac{\partial \varphi}{\partial x_k}\right\rangle$$

$$\langle D^\alpha f, \varphi\rangle = (-1)^{|\alpha|}\langle f, D^\alpha \varphi\rangle$$

$$\langle Lf, \varphi\rangle = \langle f, L^*\varphi\rangle$$

L^* is <u>formal</u> adjoint

<u>Example</u> $\quad H(x) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } x \geq 1 \end{cases}$

$$\langle H', \varphi\rangle = -\langle H, \varphi'\rangle = -\int_0^\infty \varphi'(x)\,dx = \varphi(0) = \langle \delta_0, \varphi\rangle$$

$\therefore H' = \delta_0 \qquad H_a(x) = H(x-a)$

$\qquad\qquad\qquad H_a' = \delta_a$

<u>Example</u> $f(x) = |x|$

$$\langle |x|', \varphi \rangle = -\langle |x|, \varphi' \rangle = \cdots = \int_{-\infty}^{\infty} \operatorname{sgn}(x)\, \varphi(x)\, dx$$

$$\therefore\ |x|' = \operatorname{sgn}(x) = 2H(x) - 1$$
$$|x|'' = 2\delta_0(x)$$

<u>Note</u> Distribution are infinitely often differentiable.

<u>Application</u> Weak solutions to ODE & PDE.

<u>Theorem</u> The general distributional soln (weak soln) to $u' = 0$ is $u = c$ st.

~~Th $u' = 0 \Rightarrow \langle u, \varphi' \rangle = 0\ \forall$~~

<u>Proof</u> $u' = 0$ means $\langle u, \theta' \rangle = 0\ \forall \theta \in \mathcal{D}$.
pick $\psi \in \mathcal{D}$ s.t. $\int \psi\, dx = 1$
(e.g. $S_\varepsilon(x)$ for any ε.) Take any $\varphi \in \mathcal{D}$
Let $c_\varphi = \langle 1, \varphi \rangle$

$$\theta := \int_{-\infty}^{x} [\varphi(s) - c_\varphi \psi(s)]\, ds \quad \in C_0^\infty$$

$$\therefore\ \theta' = \varphi - c_\varphi \psi \quad \therefore\ \varphi = \theta' + c_\varphi \psi$$

$$\langle u, \varphi \rangle = \langle u, \theta' \rangle + c_\varphi \langle u, \psi \rangle$$

$$= 0 + \langle u, \psi \rangle \langle 1, \varphi \rangle \quad \text{Let } c = \langle u, \psi \rangle$$

$$\langle u, \varphi \rangle = \langle c, \varphi \rangle \quad \forall \varphi \quad \therefore\ u = c$$

<u>Example</u> Let f and $g \in L_1^{loc}(\mathbb{R})$

$$\frac{\partial}{\partial x \partial y}\left[f(x) + g(y)\right] = 0 \quad \text{in weak sense}$$

<u>Ck</u> $\left\langle \frac{\partial}{\partial x \partial y}\left[f(x) + g(y)\right], \varphi(x,y) \right\rangle =$ by def

$$\left\langle f(x) + g(y), \frac{\partial^2 \varphi}{\partial y \partial x} \right\rangle$$

$$= \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} \frac{\partial^2 \varphi}{\partial y \partial x}\, dy\, dx + \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} \frac{\partial^2 \varphi}{\partial x \partial y}\, dx\, dy$$

$$= \int_{-\infty}^{\infty} f(x) \cdot 0\, dx + \int_{-\infty}^{\infty} g(y) \cdot 0\, dy = 0$$

$$= \langle 0, \varphi(x,y) \rangle \quad \forall\ \varphi \in \mathcal{D}(\mathbb{R}^2)$$