Distributions (part 2)

Tensor products of distributions

Let $f \in L_1^{loc}(\Omega_1)$ $\quad g \in L_1^{loc}(\Omega_2)$

$\varphi \in \mathcal{D}(\Omega_1 \times \Omega_2)$

$$\langle f \otimes g, \varphi \rangle = \int_{\Omega_1} f(x) \int_{\Omega_2} g(y)\, \varphi(x,y)\, dy\, dx$$

More generally $f \in \mathcal{D}(\Omega_1)'$ $\quad g \in \mathcal{D}(\Omega_2)'$

$f \otimes g \in \mathcal{D}(\Omega_1 \times \Omega_2)'$

$$\langle f \otimes g, \varphi(\cdot,\cdot) \rangle = \langle f_{(x)}, \langle g_{(y)}, \phi(x,y) \rangle \rangle$$

<u>Example</u> δ 1-dim delta distribution
$\delta \otimes \delta$ 2-dim " "

$$\langle \delta \otimes \delta, \phi(x,y) \rangle = \langle \delta, \phi(x,0) \rangle = \phi(0,0)$$

<u>Example</u> Show in $\mathbb{R}^2$ $\quad \Delta \ln r = 2\pi \delta$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad r^2 = x^2 + y^2$$

$$\langle \Delta \ln r, \phi(r,\theta) \rangle = \langle \ln r, \Delta \varphi(r,\theta) \rangle$$

$$= \int_0^{2\pi} \int_0^{\infty} \ln r \left[\frac{1}{r}(r \phi_r)_r + \frac{1}{r^2} \varphi_{\theta\theta} \right] r\, dr\, d\theta$$

Let $\Phi(r) = \int_0^{2\pi} \varphi(r,\theta)\, d\theta$ Note $\int_0^{2\pi} \phi_{\theta\theta}\, d\theta = 0$

$\therefore$ integrate first over θ :

$$\int_0^\infty \ln r \left[\frac{1}{r}(r\Phi')'\right] r\, dr$$

$$= \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty \ln r\, (r\Phi')'\, dr$$

$$= \lim_{\varepsilon \downarrow 0} \left[(\ln r)(r\Phi')\Big|_\varepsilon^\infty - \int_\varepsilon^\infty \Phi'(r)\, dr\right]$$

$$= \lim_{\varepsilon \downarrow 0} + \int_0^{2\pi} \varphi(\varepsilon,\theta)\, d\theta = 2\pi\varphi(0,\theta)$$

$$-\Delta\left(-\frac{1}{2\pi}\ln r\right) = \delta_0$$

$$-\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-\eta)^2} = s(x,y;\xi,\eta)$$

= singular part of $g(x,y;\xi,\eta)$, the

Green's function for the problem

$$-\Delta u = f(r,\theta) \quad \text{in } \Omega$$
$$+ \text{ B.C. on } \partial\Omega$$

$g(x,y;\xi,\eta) = s(x,y;\xi,\eta) + h(x,y;\xi,\eta)$ when h is a C^2 fcn s.t. g satisfies BC.

Def: Let S be set with a relation $\leq$ satisfying
(i) $x \leq x$ (ii) $x \leq y$ and $y \leq z$ implies $x \leq z$ ~~and~~
(iii) $x \leq y$ and $y \leq x$ implies $x = y$.
Then $(S, \leq)$ is called a <u>partially</u> <u>ordered</u> <u>set</u>

Def: Let $(S, \leq)$ be a partially ordered set s.t.
$\forall x \in S$ and $y \in S$ $\exists u \in S$ s.t. $x \leq u$ and $y \leq u$
then $(S, \leq)$ is called a <u>directed</u> <u>set</u>.

<u>Def:</u> A function $f : A \to X$ where
$(A, \leq)$ is a ~~p~~ directed set is called a
net on X

<u>Example</u> $A = \mathbb{N}$ $\leq$ usual order on positive integers
$f : \mathbb{N} \to X$ is a net. It is a
special net, namely a sequence
$\therefore$ nets are a generalization of sequences

<u>Def.</u> Let $(X, \mathcal{T})$ be a topological space
and $(A, \leq)$ a directed set
Let $\{f_\alpha\}_{\alpha \in A}$ be a net on X. We

say that $\lim_\alpha f_\alpha = f \in X$ if $\forall$ open set

U containing f $\exists$ $\alpha_U \in A$ s.t. $f_\alpha \in U$
$\forall$ ~~alpha~~ $\alpha_U \leq \alpha$

Note if $\alpha \leq \beta$ we also may write $\beta \geq \alpha$
$\therefore$ $f_\alpha \in U$ $\forall \alpha \geq \alpha_U$

Example $A = [0, \infty)$, $\leq$ as usual.

$f_t \in C[0,\infty)$ with the topology of pointwise convergence

$$f_t(x) = \int_0^\infty e^{-tsx} g(s)\, ds$$

where g is some bounded continuous function

$$\lim_t \left(\lim_{t \to \infty}\right) f_t = \text{zero function}$$

Definition Let $\{f_\alpha\}_{\alpha \in A}$ be a net of distributions : $f_\alpha \in \mathcal{D}(\Omega)'$ We say $\lim_\alpha f_\alpha = f$ if

$$\lim_\alpha \langle f_\alpha, \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$$

Note that f is a linear functional. It can be shown that f must also be continuous

Example Consider the regular distributions $\{\sin(kx)\}_{k \in \mathbb{N}}$

Claim $\lim_{k \to \infty} \sin(kx) = 0$ in the ~~[illegible]~~ sense of ~~s~~ distributions

Proof: $\left|\int_{-\infty}^{\infty} \sin(kx)\varphi(x)\,dx\right| = \left|\int_{-\infty}^{\infty} -\frac{d}{dx}\left(\frac{\cos(kx)}{k}\right)\varphi(x)\,dx\right|$

$= \frac{1}{k}\left|\int_{-\infty}^{\infty} \cos(kx)\,\varphi'(x)\,dx\right| \leq \frac{1}{k}\int_{-\infty}^{\infty} |\varphi'(x)|\,dx \to 0$ as $k \to \infty$.

Related to the above result is the following

<u>Riemann-Lebesgue Lemma</u> If $f \in L_1(-\infty, \infty)$ then $\int_{-\infty}^{\infty} f(x) e^{itx}\,dx$ exists and

$$\lim_{t \to \pm\infty} \int_{-\infty}^{\infty} f(x) e^{itx}\,dx = 0.$$

<u>Theorem</u> If $f_\alpha \to f$ in $\mathcal{D}(\Omega)'$ then for any multi-index γ we have $D^\gamma f_\alpha \to D^\gamma f$.

Proof: trivial.

<u>Example</u> $\delta_k(x) = \begin{cases} 0 & \text{if } |x| > 1/2k \\ k & \text{if } |x| \leq 1/2k \end{cases}$

Claim: $\lim_{k\to\infty} \delta_k = \delta$. Let $H_k(x) = \begin{cases} 0 & x < -1/2k \\ k(x+\frac{1}{2k}) & \text{if } |x| \leq \frac{1}{2k} \\ 1 & \text{if } x > 1/2k \end{cases}$

We easily see $H_k \to H$ in L_1 $\therefore \langle H_k, \varphi\rangle \to \langle H, \varphi\rangle$

Then apply the above theorem. Also this can be shown directly:

$$\int_{-\infty}^{\infty} \delta_k(x)\varphi(x)\,dx - \varphi(0) = \int_{-\infty}^{\infty} \delta_k(x)\varphi(x)\,dx - \int_{-\infty}^{\infty} \delta_k(x)\varphi(0)\,dx$$

$= \int_{-\infty}^{\infty} \delta_k(x)[\varphi(x) - \varphi(0)]\,dx$. Given ε $\exists\,\delta$ s.t. $|\varphi(x) - \varphi(0)| < \varepsilon$ if $|x - 0| < \delta$. So for $k > 1/\varepsilon$

$$\left|\int_{-\infty}^{\infty} \delta_k(x)\varphi(x)\,dx - \varphi(0)\right| \leq \int_{-\infty}^{\infty} \delta_k(x)|\varphi(x) - \varphi(0)|\,dx < \varepsilon \int_{-\infty}^{\infty} \delta_k(x) = \varepsilon.$$