Distributions (part 2)

Tensor products of distributions

Let $f \in L_1^{loc}(\Omega_1)$ $\quad g \in L_1^{loc}(\Omega_2)$

$\varphi \in \mathcal{D}(\Omega_1 \times \Omega_2)$

$$\langle f \otimes g, \varphi \rangle = \int_{\Omega_1} f(x) \int_{\Omega_2} g(y)\, \varphi(x,y)\, dy\, dx$$

More generally $f \in \mathcal{D}(\Omega_1)'$ $\quad g \in \mathcal{D}(\Omega_2)'$

$f \otimes g \in \mathcal{D}(\Omega_1 \times \Omega_2)'$

$$\langle f \otimes g, \varphi(\cdot,\cdot) \rangle = \langle f_{(x)}, \langle g_{(y)}, \phi(x,y) \rangle \rangle$$

<u>Example</u> δ 1-dim delta distribution
$\delta \otimes \delta$ 2-dim " "

$$\langle \delta \otimes \delta, \phi(x,y) \rangle = \langle \delta, \phi(x,0) \rangle = \phi(0,0)$$

<u>Example</u> Show in $\mathbb{R}^2$ $\quad \Delta \ln r = 2\pi \delta$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad r^2 = x^2 + y^2$$

$$\langle \Delta \ln r, \phi(r,\theta) \rangle = \langle \ln r, \Delta \varphi(r,\theta) \rangle$$

$$= \int_0^{2\pi} \int_0^{\infty} \ln r \left[\frac{1}{r}(r\phi_r)_r + \frac{1}{r^2} \varphi_{\theta\theta} \right] r\, dr\, d\theta$$

Let $\Phi(r) = \int_0^{2\pi} \varphi(r,\theta)\,d\theta$ Note $\int_0^{2\pi} \phi_{\theta\theta}\,d\theta = 0$

$\therefore$ integrate first over θ :

$$\int_0^\infty \ln r \left[\frac{1}{r}(r\Phi')'\right] r\,dr$$

$$= \lim_{\varepsilon\downarrow 0} \int_\varepsilon^\infty \ln r\,(r\Phi')'\,dr$$

$$= \lim_{\varepsilon\downarrow 0}\left[(\ln r)(r\Phi')\Big|_\varepsilon^\infty - \int_\varepsilon^\infty \Phi'(r)\,dr\right]$$

$$= \lim_{\varepsilon\downarrow 0} + \int_0^{2\pi} \varphi(\varepsilon,\theta)\,d\theta = 2\pi\varphi(0,\theta)$$

$$-\Delta\left(-\frac{1}{2\pi}\ln r\right) = \delta_0$$

$$-\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-\eta)^2} = s(x,y;\xi,\eta)$$

= singular part of $g(x,y;\xi,\eta)$, the

Green's function for the problem

$$-\Delta u = f(r,\theta) \quad \text{in } \Omega$$
$$+ \text{ B.C. on } \partial\Omega$$

$g(x,y;\xi,\eta) = s(x,y;\xi,\eta) + h(x,y;\xi,\eta)$ where h is a C^2 fn s.t. g satisfies BC.

Def: Let S be set with a relation $\leq$ satisfying
(i) $x \leq x$ (ii) $x \leq y$ and $y \leq z$ implies $x \leq z$ ~~and~~
(iii) $x \leq y$ and $y \leq x$ implies $x = y$.
Then $(S, \leq)$ is called a <u>partially</u> <u>ordered</u> <u>set</u>

Def: Let $(S, \leq)$ be a partially ordered set s.t.
$\forall x \in S$ and $y \in S$ $\exists u \in S$ s.t. $x \leq u$ and $y \leq u$
then $(S, \leq)$ is called a <u>directed</u> <u>set</u>.

<u>Def:</u> A function $f : A \to X$ where $(A, \leq)$ is a ~~p~~ directed set is called a net on X

<u>Example</u> $A = \mathbb{N}$ $\leq$ usual order on positive integers
$f : \mathbb{N} \to X$ is a net. It is a special net, namely a sequence
$\therefore$ nets are a generalization of sequences

<u>Def.</u> Let $(X, \mathcal{T})$ be a topological space and $(A, \leq)$ a directed set
Let $\{f_\alpha\}_{\alpha \in A}$ be a net on X. We say that $\lim_\alpha f_\alpha = f \in X$ if $\forall$ open set U containing f $\exists$ $\alpha_U \in A$ s.t. $f_\alpha \in U$
$\forall$ ~~alpha~~ $\alpha_U \leq \alpha$

Note if $\alpha \leq \beta$ we also may write $\beta \geq \alpha$
$\therefore$ $f_\alpha \in U$ $\forall \alpha \geq \alpha_U$

Example $A = [0, \infty)$, $\leq$ as usual.

$f_t \in C[0,\infty)$ with the topology of pointwise convergence

$$f_t(x) = \int_0^\infty e^{-tsx} g(s)\, ds$$

where g is some bounded continuous function

$$\lim_t \left(\lim_{t \to \infty}\right) f_t = \text{zero function}$$

Definition Let $\{f_\alpha\}_{\alpha \in A}$ be a net of distributions: $f_\alpha \in \mathcal{D}(\Omega)'$ We say $\lim_\alpha f_\alpha = f$ if

$$\lim_\alpha \langle f_\alpha, \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$$

Note that f is a linear functional. It can be shown that f must also be continuous

Example Consider the regular distributions $\{\sin(kx)\}_{k \in \mathbb{N}}$

Claim $\lim_{k \to \infty} \sin(kx) = 0$ in the ~~[illegible]~~ sense of ~~$\mathcal{S}$~~ distributions

Proof: $\left|\int_{-\infty}^{\infty} \sin(kx)\varphi(x)\,dx\right| = \left|\int_{-\infty}^{\infty} -\frac{d}{dx}\left(\frac{\cos(kx)}{k}\right)\varphi(x)\,dx\right|$

$= \frac{1}{k}\left|\int_{-\infty}^{\infty} \cos(kx)\,\varphi'(x)\,dx\right| \le \frac{1}{k}\int_{-\infty}^{\infty}|\varphi'(x)|\,dx \to 0$ as $k\to\infty$.

Related to the above result is the following

Riemann-Lebesgue Lemma If $f \in L_1(-\infty,\infty)$ then $\int_{-\infty}^{\infty} f(x)e^{itx}\,dx$ exists and

$$\lim_{t\to\pm\infty}\int_{-\infty}^{\infty} f(x)e^{itx}\,dx = 0.$$

Theorem If $f_\alpha \to f$ in $\mathcal{D}(\Omega)'$ then for any multi-index γ we have $D^\gamma f_\alpha \to D^\gamma f$.

Proof: trivial.

Example $\delta_k(x) = \begin{cases} 0 & \text{if } |x| > 1/2k \\ k & \text{if } |x| \le 1/2k \end{cases}$

Claim: $\lim_{k\to\infty}\delta_k = \delta$. Let $H_k(x) = \begin{cases} 0 & x < -1/2k \\ k(x+\frac{1}{2k}) & \text{if } |x| \le \frac{1}{2k} \\ 1 & \text{if } x > 1/2k \end{cases}$

We easily see $H_k \to H$ in L_1 $\therefore \langle H_k,\varphi\rangle \to \langle H,\varphi\rangle$

Then apply the above theorem. Also this can be shown directly:

$\int_{-\infty}^{\infty}\delta_k(x)\varphi(x)\,dx - \varphi(0) = \int_{-\infty}^{\infty}\delta_k(x)\varphi(x)\,dx - \int_{-\infty}^{\infty}\delta_k(x)\varphi(0)\,dx$

$= \int_{-\infty}^{\infty}\delta_k(x)[\varphi(x)-\varphi(0)]\,dx$. Given ε $\exists\,\delta$ s.t. $|\varphi(x)-\varphi(0)| < \varepsilon$ if $|x-0|<\delta$. So for $k > 1/\varepsilon$

$\left|\int_{-\infty}^{\infty}\delta_k(x)\varphi(x)\,dx - \varphi(0)\right| \le \int_{-\infty}^{\infty}\delta_k(x)|\varphi(x)-\varphi(0)|\,dx < \varepsilon\int_{-\infty}^{\infty}\delta_k(x) = \varepsilon.$

Delta sequences and delta nets

A delta sequence is a sequence $\{\delta_k\}$ of integrable functions which converge in the distributional sense to δ, the delta distribution:

$$\int f(x)\,\delta_k(x)\,dx \longrightarrow f(0)$$

We restrict ourselves to sequences. The generalization to nets is simple and obvious. We will consider two types of delta sequences: those of positive type and those of Dirichlet type

Definition. A sequence $\{\delta_k\}$ of integrable functions with support contained in a closed bounded interval $[a,b]$ is said to be a delta sequence of positive type if

(i) $\displaystyle\lim_{k\to\infty}\int_{\mathbb{R}}\delta_k(x)\,dx = 1$

(ii) $\forall c>0$ $\delta_k \to 0$ uniformly on $\mathbb{R}\setminus(-c,c)$

(iii) $\delta_k(x)\geq 0$ $\forall x$.

Simple examples are

$$\delta_k(x) = \begin{cases} k & \text{if } |x|\leq \frac{1}{2k} \\ 0 & \text{if } |x|>\frac{1}{2k}\end{cases}$$

$$\delta_k(x) = \left\{\int_{-\infty}^{\infty} e^{-k^2x^2}\,dx\right\}^{-1} e^{-k^2x^2}\chi(x) \qquad \chi(x) = \begin{cases} 1 & \text{if } |x|\leq 1 \\ 0 & \text{if } |x|>1\end{cases}$$

<u>Definition</u> A sequence $\{\delta_k\}$ of integrable functions with support contained in a closed bounded interval $[a,b]$ is said to be a <u>delta sequence</u> <u>of</u> <u>Dirichlet</u> <u>type</u> if

$$\text{(i)} \quad \lim_{k\to\infty} \int_{\mathbb{R}} \delta_k(x)\,dx = 1$$

$$\text{(ii)} \quad \forall c>0, \ \forall h \in L_1^{loc}(\mathbb{R})$$

$$\lim_{k\to\infty} \int_{\mathbb{R}\setminus(-c,c)} h(x)\,\delta_k(x)\,dx = 0$$

$$\text{(iii)} \quad \exists A>0 \text{ s.t. } |\delta_k(x)| \le \frac{A}{|x|} \quad \text{if } |x|>0$$

Theorem (pointwise convergence) Let $\{\delta_k\}$ be a delta sequence and $f \in L_1(\mathbb{R})$ then

$$\gamma_k := \int_{\mathbb{R}} f\,\delta_k(x)\,dx \longrightarrow f(0) \quad \text{as } k\to\infty \text{ if}$$

(i) δ_k is of positive type and f is continuous at 0

OR

(ii) δ_k is of Dirichlet type and $f'(0)$ exists.

<u>Proof</u>: There exists k_0 s.t. $\int \delta_k(x)\,dx \le 2 \ \forall k \ge k_0$

If $f'(0)$ exists, $\exists\ \eta > 0$ and $M > 0$ s.t. $\left|\frac{f(x)-f(0)}{x}\right|$ $\le M$ if $|x| < \eta$. Let us define

$$I_1 := \int_{-c}^{c} f\,\delta_k\,dx \qquad I_2 = \int_{\mathbb{R}\setminus(-c,c)} f\,\delta_k\,dx$$

$$I_1 = f(0)\int_{-c}^{c} \delta_k\,dx + \int_{-c}^{c} [f(x) - f(0)]\,\delta_k\,dx$$
$$\equiv I_1' + I_1''$$

Given $\varepsilon > 0$ we choose c as follows:
Under hypotheses (i) $\exists\ \delta$ s.t. $|f(x) - f(0)| < \varepsilon$ if $|x| < \delta$. Choose $c < \delta$, then

$$|I_1''| \le \int_{-c}^{c} \varepsilon\,\delta_k \le \varepsilon \int_{\mathbb{R}} \delta_k \le 2\varepsilon \text{ if } k \ge k_0$$

Under hypothesis (ii) we choose $c < \varepsilon/(MA)$:

$$|I_1''| \le \int_{-c}^{c} \left|\frac{f(x)-f(0)}{x}\right| A\,dx \le \int_{-c}^{c} MA\,dx = 2MAc < 2\varepsilon$$

So c is now fixed and $k \ge k_0$.
Clearly, for k sufficiently large $|I_2| < \varepsilon$
Note that

$$\left|1 - \int_{-c}^{c} \delta_k\,dx\right| \le \left|1 - \int_{\mathbb{R}} \delta_k\,dx\right| + \left|\int_{\mathbb{R}\setminus(-c,c)} \delta_k\,dx\right|$$

$< \varepsilon$ for k sufficiently large

Therefore, for k sufficiently large

$$|f(0) - \gamma_k| \leq |f(0) - I_1'| + |I_1''| + |I_2|$$

$$< |f(0)|\varepsilon + 2\varepsilon + \varepsilon$$

Hence $\lim_{k \to \infty} \gamma_k = f(0)$

Jump discontinuities

Suppose $\delta_k(x) = \delta_k(-x)$ then

$\int_0^\infty \delta_k(x)\,dx \to 1/2$ and hence if

f is the Heaviside function f_H

$$\int_{-\infty}^{\infty} f_H(x)\,\delta_k(x)\,dx \longrightarrow \frac{1}{2} = \frac{f_H(0+) + f_H(0-)}{2}$$

If f has a jump discontinuity at 0 but $f(0+)$ and $f(0-)$ exist and δ_k is symmetric of positive type then

$$\int f(x)\,\delta_k(x)\,dx \longrightarrow \frac{f(0+) + f(0-)}{2}$$

The same is true if δ_k is of Dirichlet type but we will also need the existence of one-sided derivatives $f'(0+)$ and $f'(0-)$

Uniform Convergence

Def. The convolution of f and g

$$h(x) = \int_{-\infty}^{\infty} f(x-t)\,g(t)\,dt = \int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt \qquad h = f * g$$

An inspection of the proof of the pointwise convergence theorem shows that $|\delta_k - f(0)|$ depends on k, the size of $|f(0)|$ and the modulus of continuity at $x=0$. Therefore we have

Theorem Suppose that f is a bounded uniformly continuous function on $\mathbb{R}$. then
$\delta_k * f \to f$ uniformly

Note: $(\delta_k * f)(x) = \int f(t)\,\delta_k(x-t)\,dt = \int f(x-\delta)\,\delta_k(t)\,dt$

$$\to f(x) \text{ as } k \to \infty.$$

Applications

Weierstrass Approximation Theorem. Let $f \in C[\alpha,\beta]$ then $\exists$ a sequence of polynomials, $\{p_k\}$ s.t. $p_k \to f$ uniformly on $[\alpha,\beta]$ as $k \to \infty$.

Proof: Let $b>0$, to be chosen later, and define

$$\delta_k(x) = \begin{cases} A_k^{-1}(b^2-x^2)^k & \text{if } |x|<b \\ 0 & \text{if } |x|\geq b \end{cases}$$

where $A_k = \int_{-b}^{b}(b^2-x^2)^k\,dx$. Clearly $\delta_k(x)\geq 0$

and $\int_{\mathbb{R}}\delta_k(x)\,dx = 1$. Pick any $c\in(0,b)$ then

$$A_k = \int_{-b}^{b}(b^2-x^2)^k\,dx \geq \int_{-c/2}^{c/2}(b^2-x^2)^k\,dx \geq c\,(b^2-c^2/4)^k$$

and then, if $|x|\geq c$, $\delta_k(x)\leq\delta_k(c) = A_k^{-1}(b^2-c^2)^k$

$$\leq \frac{1}{c}\left(\frac{b^2-c^2}{b^2-c^2/4}\right)^k = \rho^k/c \quad \text{where } \rho=(b^2-c^2)/(b^2-\tfrac{c^2}{4})<1.$$

Hence $\delta_k(x)\to 0$ uniformly in $\mathbb{R}\setminus(-c,c)$. Let F be an extension of f to a (uniformly) continuous function of compact support and let $K>0$ s.t. $\operatorname{supp}(F)\subset[-K,K]$. Next choose b so large that $b>\beta+K$ and $b>K-\alpha$ and define $p_k=\delta_k * F$

$$p_k(x) = \int_{-\infty}^{\infty}F(t)\,A_k^{-1}\left(b^2-(x-t)^2\right)^k dt \qquad (\alpha\leq x\leq\beta)$$

$$= \int_{-K}^{K}F(t)\,A_k^{-1}\left(b^2-(x-t)^2\right)^k dt.$$

Clearly p_k is a polynomial and $p_k\to F$ uniformly on $[\alpha,\beta]$. Note $p_k(x)$ is not a polynomial for all x, but is a polynomial on $[\alpha,\beta]$.

<u>Definition</u> We say that f is a piecewise continuous function on $[\alpha,\beta]$ if $\exists$ finitely many pts γ_i
$\gamma_0 = \alpha < \gamma_1 < \gamma_2 < \cdots < \gamma_N < \beta = \gamma_{N+1}$ s.t. f is continuous on each subinterval (γ_i, γ_{i+1}) and $\lim_{x\downarrow\gamma_i} f(x)$ $\exists$ for $i = 0,1,\ldots,\gamma_N$ and $\lim_{x\uparrow\gamma_i} f(x)$ $\exists$ for $i = 1,2,\ldots,\gamma_{N+1}$

<u>Definition</u> We say that f is piecewise smooth on $[\alpha,\beta]$ if f and f' are piecewise continuous.

<u>Remark</u> If f is piecewise continuous on $[\alpha,\beta]$ then, as in the proof of the Weierstrass approximation theorem we can construct the polynomials p_k but

$p_k(x) \longrightarrow \frac{1}{2}\left(f(x+) + f(x-)\right)$ as $k \to \infty$

<u>Example</u> (Dirichlet Kernel) for $-\pi \le x \le \pi$ let

$$\delta_k(x) = \frac{1}{\pi}\left(\frac{1}{2} + \cos x + \cos(2x) + \cdots + \cos(kx)\right)$$

$$= \frac{\sin\left((k+\frac{1}{2})x\right)}{2\pi \sin(x/2)} \qquad k = 0,1,2,\ldots$$

For $|x| > \pi$ we can set $\delta_k(x)$

$$\int \delta_k(x)\,dx = \int_{-\pi}^{\pi} \delta_k(x)\,dx = \int_{-\pi}^{\pi} \frac{1}{2}\,dx + 0 = 1$$

since $\int_{-\pi}^{\pi} \cos(jx)\,dx = 0$ for $j = 1,2,\ldots$

If $h \in L_1^{loc}(\mathbb{R})$ and $c > 0$ then

$$\int_{\mathbb{R}\setminus(-c,c)} h\,\delta_k(x)\,dx = \left(\int_{-\pi}^{-c} + \int_c^{\pi}\right) \frac{h}{\sin(x/2)} \sin((k+\tfrac{1}{2})x)\,dx$$

where $\dfrac{h}{\sin(x/2)} \in L_1((-\pi,-c)\cup(c,\pi))$. Hence, by the Riemann-Lebesgue lemma $\lim_{k\to\infty} \int_{\mathbb{R}\setminus(-c,c)} h\,\delta_k\,dx = 0$.

We also note that

$$|\delta_k(x)| \le \frac{1}{2\pi \sin(x/2)} \quad \text{for } -\pi \le x \le \pi$$

$|\sin\frac{x}{2}| \ge \frac{1}{\pi}|x| \quad \therefore \quad |\delta_k(x)| \le \frac{1}{2|x|}$

Hence $\{\delta_k\}$ is a Dirichlet type delta sequence
$\delta_k(x-y)$ is called the Dirichlet Kernel

Application: Pointwise convergence of Fourier series.

We already saw that $\{e^{ikx}\}_{k=-\infty}^{\infty}$ forms a basis for $L_2(-\pi,\pi)$ and that $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$

converges to f in the $L_2(-\pi,\pi)$ norm provided $f \in L_2(-\pi,\pi)$ and $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f e^{-ikx}\,dx$. Now we will study pointwise and uniform convergence of Fourier series. First we assume $f \in C'(\mathbb{R})$ and is 2π-periodic.

$$f_k(x) := \sum_{k=N}^{N} c_k e^{ikx} =$$

$$\sum_{-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ik(x-t)}f(t)\,dt = \int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\sum_{-N}^{N}e^{ik(x-t)}\right)f(t)\,dt$$

$$= \int_{-\pi}^{\pi}\frac{1}{\pi}\left(\frac{1}{2}+\cos(x-t)+\cos 2(x-t)+\cdots+\cos N(x-t)\right)f(t)\,dt$$

$$= \int_{-\pi}^{\pi}\frac{\sin\left((k+\frac{1}{2})(x-t)\right)}{2\pi\sin((x-t)/2)}\,f(t)\,dt$$

By the fact that the integrand is 2π-periodic

$$= \int_{-\pi}^{\pi}\frac{\sin\left((k+\frac{1}{2})t\right)}{2\pi\sin(t/2)}\,f(x-t)\,dt$$

$$= \int_{-\pi}^{\pi}\delta_N(t)\,f(x-t)\,dt \longrightarrow f(x) \text{ as } N\to\infty$$

Since $\delta_N(t)=\delta_N(-t)$ we can extend this result to piecewise smooth functions:

Theorem Let f be a piecewise smooth 2π-periodic function. Then its Fourier series $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$, $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)e^{-iks}\,ds$ converges pointwise to $\frac{1}{2}(f(x+)+f(x-))$

$$\sum_{k=-N}^{N} c_k e^{ikx} \longrightarrow \frac{f(x+)+f(x-)}{2} \quad \text{as } \cancel{k}\, N\to\infty$$

If f is, in addition to being piecewise smooth, also continuous then it is not too difficult to show that the Fourier series converges uniformly

Fourier Transforms

Def $f: \mathbb{R} \to \mathbb{C}$ is said to be piecewise continuous (resp. piecewise smooth) if f is piecewise continuous (resp. piecewise smooth) on any bounded interval.

If $f \in L_1(\mathbb{R})$ then we define its Fourier transform

$$\mathcal{F}(f) = \hat{f} \qquad \hat{f}(\omega) = \int_{-\infty}^{\infty} f(\xi) e^{i\omega\xi}\, d\xi$$

Often we use the corresponding capital letters to denote Fourier transforms; $\mathcal{F}(f) = F$, $\mathcal{F}(g) = G$, etc.

Theorem Suppose $f \in L_1(\mathbb{R})$ and is piecewise smooth then we may define the inverse Fourier transform of $F = \mathcal{F}(f)$ by

$$\mathcal{F}^{-1}(F) = \tilde{F} \qquad \tilde{F}(x) = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega$$

where PV stands for Cauchy Principal Value i.e.

$$\tilde{F}(x) = \frac{1}{2\pi} \lim_{R \to \infty} \int_{-R}^{R} F(\omega) e^{-i\omega x}\, d\omega$$

Moreover

$$\lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^{R} F(\omega) e^{-i\omega x}\, d\omega = \frac{f(x+) + f(x-)}{2}$$

$\left(\mathcal{F}^{-1}(\mathcal{F}(f))\right)(x) = f(x)$ at all points of continuity.

Proof:

$$\tilde{F}(x) = \frac{1}{2\pi} \lim_{R\to\infty} \int_{-R}^{R} \int_{-\infty}^{\infty} f(\xi) e^{i\omega(\xi - x)} \, d\xi \, d\omega$$

By Fubini's theorem we can interchange order of integration

$$= \lim_{R\to\infty} \int_{-\infty}^{\infty} f(\xi) \frac{1}{2\pi} \int_{-R}^{R} e^{i\omega(\xi - x)} \, d\omega \, d\xi$$

$$= \lim_{R\to\infty} \int_{-\infty}^{\infty} f(\xi) \frac{1}{\pi} \int_{0}^{R} \cos \omega(\xi - x) \, d\omega \, d\xi$$

$$= \lim_{R\to\infty} \int_{-\infty}^{\infty} f(\xi) \left. \frac{\sin \omega(\xi - x)}{\pi(\xi - x)} \right|_{\omega=0}^{\omega=R} d\xi$$

$$= \lim_{R\to\infty} \int_{-\infty}^{\infty} f(\xi) \, \delta_R(x - \xi) \, d\xi$$

where

$$\delta_R(s) = \frac{\sin(Rs)}{\pi s}$$

We only need to verify that $\{\delta_R\}$ is a delta net.

$$\int_{-\infty}^{\infty} \frac{\sin(Rs)}{\pi s} \, ds = \int_{-\infty}^{\infty} \frac{\sin \tau}{\pi \tau} \, d\tau = 1$$

By the Riemann-Lebesgue theorem $\int_{\mathbb{R}\setminus(-c,c)} \delta_R(s) \, h(s) \, ds \to 0$

as $R \to \infty$ for any $h \in L_1(-\infty, \infty)$. Finally,
$|\sin(Rs)/\pi s| \le 1/\pi|s|$.