MAT 551 Fall 07 HW set 3 Answers

1. This problem is like #5 on HW set 2

2a i) $\|L\| = \sup_{\|x\|=1} \|Lx\| \geq 0$. If $\|L\|=0 \Rightarrow \|Lx\|=0$ for all x $\therefore$ $Lx = 0$ for all x $\therefore$ $L=0$

ii) $\|cL\| = \sup_{\|x\|=1} \|cLx\| = \sup_{\|x\|=1} |c|\,\|Lx\| = |c| \sup_{\|x\|=1} \|Lx\| = |c|\,\|L\|$

iii) $\|L+M\| = \sup_{\|x\|=1} \|(L+M)x\| = \sup_{\|x\|=1} \left(\|Lx\| + \|Mx\|\right)$

$\leq \sup_{\|x\|=1} \|Lx\| + \sup_{\|x\|=1} \|Mx\|$. The last <u>inequality</u> follows from the fact that $\sup\{\alpha+\beta \mid \alpha \in A, \beta \in B\} \leq \sup\{\alpha \mid \alpha \in A\} + \sup\{\beta \mid \beta \in B\}$

2b. Let $\{L_i\}_{i=1}^{\infty}$ be a Cauchy sequence in $\mathcal{B}(X,Y)$ then for each x $\{L_i x\}_{i=1}^{\infty}$ is a Cauchy sequence in Y. Completeness of Y implies

$$Lx := \lim_{i\to\infty} L_i x \quad \text{exists } \forall x \in X$$

L is linear from standard results on limits. Since $\{L_i\}_{i=1}^{\infty}$ is Cauchy in norm, $\sup_i \|L_i\| < \infty$ $\therefore$ $\|L\| < \infty$.

Given $\varepsilon > 0$ $\exists$ N s.t. $\|L_i - L_j\| < \varepsilon/2$ if $i,j \geq N$

Let $\|x\|=1$ $\|Lx - L_j x\| \leq \|L_i x - Lx\| + \|(L_i - L_j)x\|$

For i sufficiently large $\|L_i x - Lx\| < \varepsilon/2$ and $i \geq N$.

So $\|Lx - L_j x\| < \varepsilon$ if $j \geq N$ $\forall$ x with $\|x\|=1$

$\therefore$ $\|L - L_j\| < \varepsilon$ if $j \geq N$.

3a Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

then $\langle L\vec{x}, \vec{x} \rangle = (x, y)\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x, y)\begin{pmatrix} x+y \\ -x+y \end{pmatrix}$

$= x^2 + y^2 \geq 0 \quad \forall \begin{pmatrix} x \\ y \end{pmatrix}$ $\therefore$ L is positive.

But clearly $L \neq L^*$ since $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^* = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

3b. Let $B = \dfrac{A+A^*}{2}$ $\quad C = \dfrac{A-A^*}{2i}$ then

$B = B^*$ $\quad C = C^*$ and $A = B + iC$

$\langle Ax, x \rangle = \langle Bx, x \rangle + i\langle Cx, x \rangle$

If $\langle Ax, x \rangle \geq 0 \ \forall x \Rightarrow \langle Cx, x \rangle = 0 \ \forall x$

$\therefore$

$$0 = \langle C(x+y), x+y \rangle = \langle Cx, x \rangle + \langle Cx, y \rangle + \langle Cy, x \rangle + \langle Cy, y \rangle$$

$\therefore \quad 0 = \langle Cx, y \rangle + \langle Cy, x \rangle \qquad (1)$

Replace y by iy :

$$0 = -i\langle Cx, y \rangle + i\langle Cy, x \rangle$$

$\therefore$

$0 = -\langle Cx, y \rangle + \langle Cy, x \rangle \qquad (2)$

Add (1) + (2) $\Rightarrow$ $2\langle Cy, x \rangle = 0 \ \forall x, y$

$\therefore \langle Cy, x \rangle = 0 \ \forall x, y \Rightarrow C = 0$

$\therefore A = B = \frac{A+A^*}{2}$ Therefore $A = A^*$

4. Let H be a Hilbert space and $\{\vec{e}_1, \vec{e}_2, \ldots\}$ an orthonormal basis for H

Define $a_{ij} = \langle L\vec{e}_j, \vec{e}_i \rangle \quad \therefore Le_j = \sum_i a_{ij} e_i$

If $\vec{x} = \sum_i x_i \vec{e}_i \qquad L\vec{x} = \vec{y} = \sum_j y_j \vec{e}_j$

$$L\vec{x} = \sum_i x_i \sum_k a_{ki} e_k = \sum_k \left(\sum_i a_{ki} x_i \right) e_k$$

$$= \vec{y} = \sum_k y_k \vec{e}_k \qquad \therefore \quad y_k = \sum a_{ki} x_i$$

$$\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \end{pmatrix}$$

$$a^*_{ij} = \langle L^* \vec{e}_j, \vec{e}_i \rangle = \langle \vec{e}_j, L\vec{e}_i \rangle = \overline{\langle L\vec{e}_i, \vec{e}_j \rangle} = \overline{a_{ji}}$$

5. $\vec{x} = \sum_i x_i \vec{e}_i \qquad \|\vec{x}\|^2 = \sum_i |x_i|^2$

$$\|\vec{x} - P_m \vec{x}\|^2 = \sum_{m+1}^{\infty} |x_i|^2 = \|\vec{x}\|^2 - \sum_{i=1}^{m} |x_i|^2 \longrightarrow 0 \text{ as } m \to \infty$$

~~$P_m x - x$~~ $\|P_m e_k - e_k\| = 1$ if ~~e_k~~ $k > m$

$\therefore \|P_m - I\| \geq 1 \qquad \forall m$

6. $\langle T_n x, y\rangle \to f(x,y)$ as $n\to\infty$
We easily see that $f(x,y)$ is bilinear

Fix x. $y \longrightarrow \langle y, T_n x\rangle$ are bounded linear functionals so at each y $\{|\langle y, T_n x\rangle|\}_{n=1}^{\infty}$ is bounded. By the uniform boundedness theorem $\{\|T_n x\|\}_{n=1}^{\infty}$ are bounded by a constant (depending on x, of course)

Applying the uniform boundedness theorem once again we see that $\{\|T_n\|\}_{n=1}^{\infty}$ is a bounded set. i.e. $\|T_n\| \le M$ $\therefore |\langle T_n x, y\rangle| \le M\|x\|\|y\|$

$$\therefore \quad |f(x,y)| \le M\|x\|\|y\|$$

by the representation theorem for bounded bilinear functionals:

$$|f(x,y)| \le M\|x\|\|y\| \Rightarrow \exists\ T \in \mathcal{B}(H)$$

such that $f(x,y) = \langle Tx, y\rangle$

$\therefore$ $T_n \longrightarrow T$ in the weak operator topology

7a Let $U_{T_0 x y \varepsilon}$ be the weak neighborhood of T_0

defined by $\{T \,|\, |\langle (T-T_0)x, y\rangle| < \varepsilon\}$

$T^* \in U_{S^* x y \varepsilon}$ if

$$|\langle (T^* - S^*)x, y\rangle| < \varepsilon$$

This is true if $|\langle (T-S)y, x\rangle| < \varepsilon$

i.e. if $T \in U_{S y x \varepsilon}$

b) Let H be a Hilbert space with orthonormal basis $\{e_1, e_2, \dots\}$ and let P_m be the projection onto the span of $\{e_1, e_2, \dots, e_m\}$

$$L_m = P_m + \tfrac{1}{m}(I - P_m)$$

$$L_m^{-1} = P_m + m(I - P_m)$$

$$\|L_m^{-1}\| = m \longrightarrow \infty$$

$$\|L_m x - Ix\| \leq \|P_m x - x\| + \tfrac{1}{m}\|x\| \to 0$$

as $m \to \infty$ $\therefore$ $L_m \to I$ in the strong topology

$L_m^{-1} \not\to I$ in the strong topology (if it did, then by the uniform boundedness theorem $\|L_m\|$'s would have to be bounded

Let $x = \sum_{k=1}^{\infty} \frac{1}{k} e_k$ OR $L_m^{-1}x - x = \sum_{k=m+1}^{\infty} \frac{m}{k} e_k$

$$\|L_m^{-1}x - x\|^2 = \sum_{k=m+1}^{\infty} \left(\frac{m}{k}\right)^2 \ge \left(\frac{m}{m+1}\right)^2 \ge \frac{1}{4} \ \forall m$$

8.a) If $u \in L_2(0,a)$ let $\tilde{u}(x) = \begin{cases} u(x) & \text{if } 0 \le x \le a \\ 0 & \text{elsewhere} \end{cases}$

$\tilde{u} \in L_2(\mathbb{R})$ Let $\tilde{v}(x) = \begin{cases} 1/\sqrt{x} & \text{if } 0 \le x \le a \\ 0 & \text{elsewhere} \end{cases}$

then $Lu = \tilde{u} * \tilde{v}$

$\therefore \|Lu\|_2 \le \|\tilde{u}\|_2 \, \|\tilde{v}\|_1$ since $\frac{1}{2} = \frac{1}{2} + \frac{1}{1} - 1$

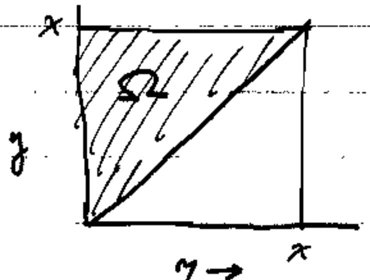
$$\|\tilde{v}\|_1 = \int_0^a \frac{1}{\sqrt{x}} = 2\sqrt{x}\Big|_0^a = 2\sqrt{a}$$

$$\therefore \|L\| \le 2\sqrt{a} < \infty.$$

b) This can be done by direct computation:

$$(A^2u)(x) = \int_0^x \frac{1}{\sqrt{x-y}} \int_0^y \frac{u(\eta)}{\sqrt{y-\eta}} \, d\eta \, dy$$

$$= \int_0^x \int_0^y \frac{u(\eta)}{\sqrt{x-y}\,\sqrt{y-\eta}} \, d\eta \, dy$$



$$= \iint_\Omega \frac{u(\eta)}{\sqrt{x-y}\,\sqrt{y-\eta}} \, dA$$

$$= \int_0^x \int_\eta^x \frac{u(\eta)}{\sqrt{x-y}\,\sqrt{y-\eta}} \, dy \, d\eta$$

$$= \int_0^x K(x,\eta)\, u(\eta)\, d\eta$$

where $K(x,\eta) = \int_\eta^x \frac{1}{\sqrt{x-y}\sqrt{y-\eta}}\, dy$

This integral can be computed (e.g. use Maple!) to be π

$$\therefore\ (A^2 u)(x) = \pi \int_0^x u(\eta)\, d\eta$$

This operator was shown to be compact in class.

More quickly we can get this result by noting that $A^{-1} = \frac{1}{\pi} A D \qquad D = \frac{d}{dx}$

$\therefore\ A^2 = \pi D^{-1} \qquad D^{-1}$ is integration

and, because $(Au)(0) = 0$ we must have

$D^{-1} f = \int_0^x f(s)\, ds$

$$\therefore\ A^2 = \pi \int_0^x dx$$

c) $$\langle Au, w \rangle = \int_0^a \bar{w}(y) \int_0^y \frac{u(\eta)}{\sqrt{y-\eta}}\, d\eta\, dy$$

$$= \int_0^a u(\eta) \int_\eta^a \frac{\bar{w}(y)}{\sqrt{y-\eta}}\, dy\, d\eta = \langle u, A^* w \rangle$$

$$\therefore\ A^* w = \int_x^a \frac{w(y)}{\sqrt{y-x}}\, dy$$