MAT 551
Fall 2007
Homework set 3

1. Let $\mathcal{S}$ be a collection of subsets of X whose union is X and such that $\emptyset \in \mathcal{S}$. Let $\mathcal{B}$ be the collection of all finite intersections of members of $\mathcal{S}$ and let $\mathcal{T}$ be the collection of arbitrary unions of members of $\mathcal{B}$. Show that $\mathcal{T}$ is a topology on X.

2. Let $\mathcal{L}(X, Y)$ be the linear space of all linear transformations from the normed space X with norm $\| \quad \|_X$ to the normed space Y with norm $\| \quad \|_Y$ and define

$$\|L\| = \sup\{\|Lx\|_Y / \|x\|_X \;:\; x \in X,\; x \neq 0\}.$$

   (a) Show that this defines a norm on $\mathcal{L}(X, Y)$.

   (b) Show that if X and Y are Banach spaces then, with this norm, $\mathcal{L}(X, Y)$ is complete.

3. Show that if the Hilbert space $\mathcal{H}$ is over the field $\mathbb{R}$ then a bounded positive operator is not necessarily self-adjoint (i.e. give a simple counterexample). But then show that if the Hilbert space $\mathcal{H}$ is over the field $\mathbb{C}$ then a bounded positive operator is always self-adjoint.

4. Let (a_{ij}) be the matrix representation of $A \in \mathcal{B}(\mathcal{H})$ and let (a^*_{ij}) be the matrix representation of A^*. Show that $a^*_{ij} = \overline{a_{ji}}$.

5. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_1, e_2, \cdots\}$ and let P_n be the projection operator onto the span of $\{e_1, e_2, \cdots, e_n\}$. Show that this sequence of operators converges to the identity operator I in the strong sense. Also $\|P_n\| = \|I\| = 1$ for all n. Show that the sequence does not converge in the uniform topology.

6. Show that if the sequence of bounded linear operators $\{T_n\}$ on a Hilbert space has the property that the numerical sequences $\{\langle T_n x, y\rangle\}$ all converge, then there is a bounded linear operator T such that $\{T_n\}$ converges weakly to T.

7. (a) Prove the map $A \to A^*$ is continuous on $\mathcal{B}(\mathcal{H})$ in the weak topology.

   (b) Let $\mathcal{GL}(\mathcal{H})$ denote the invertible bounded linear operators on $\mathcal{H}$. Prove the following map is not continuous from $\mathcal{GL}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$ in the strong topology: $A \to A^{-1}$.

8. The Abel transform, A, is defined as follows:

$$Au = f \text{ if } f(y) = \int_0^y \frac{u(\eta)}{\sqrt{y - \eta}}\, d\eta.$$

   It can be seen that this map can be inverted, at least on continuously differentiable functions:

$$u(x)) = \int_0^x \frac{f'(y)}{\pi\sqrt{x - y}}\, dy.$$

   (a) Use the inequality $\|f * g\|_r \leq \|f\|_p \, \|g\|_q$ ($1 \leq p, q, r < \infty$ with $1/r = 1/p + 1/q - 1$) to show that A is a bounded linear operator on $L_2(0, a)$ for any $0 < a < \infty$.

   (b) Show that A^2 is a compact linear operator on $L_2[0, \pi]$. (Actually also for $L_2[0, a]$ for any $0 < a < \infty$.)

   (c) Find A^* for $A : L_2[0, a] \to L_2[0, a]$