MAT 551 fall 07 HW set 1 answers (outline)

1. Let $\vec{x}_m = (1, \frac{1}{2}, \ldots, \frac{1}{m}, 0, 0, \ldots)$ then $\{\vec{x}_m\}_{m=1}^{\infty}$ is Cauchy, but it converges to $(1, \frac{1}{2}, \frac{1}{3}, \ldots) \notin V$
$\therefore$ V is not complete.

2. Suppose $\langle x_m, y\rangle \to \langle x, y\rangle$ $\forall y$ and $\|x_m\| \to \|x\|$.
Then
$\|x_m - x\|^2 = \langle x_m - x, x_m - x\rangle = \|x_m\|^2 - \langle x_m, x\rangle - \langle x, x_m\rangle + \|x\|^2$
$\to \|x\|^2 - \|x\|^2 - \|x\|^2 + \|x\|^2 = 0$
Conversely, suppose $x_n \to x$ in the strong topology ie. $\|x_n - x\| \to 0$. Then $|\|x_m\| - \|x\|| \le \|x_m - x\| \to 0$ and
$|\langle x_n, y\rangle - \langle x, y\rangle| = |\langle x_n - x, y\rangle| \le \|x_n - x\| \|y\| \to 0$.

3. $u_1 = \frac{1}{\sqrt{2}}$ $\quad u_2 = \sqrt{\frac{3}{2}}\, x$ $\quad u_3 = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$

4. a) Let $x \in N^\perp \Rightarrow \langle x, y\rangle = 0$ $\forall y \in N \supset M$
$\therefore \langle x, y\rangle = 0$ $\forall y \in M$ $\therefore x \in M^\perp$
Hence $M^\perp \supset N^\perp$

b) Let $x \in M \Rightarrow \langle x, y\rangle = 0$ $\forall y \in M^\perp$ $\therefore x \perp M^\perp$
$\therefore x \in (M^\perp)^\perp$ $\therefore$ Hence $M \subset (M^\perp)^\perp$

Conversely, suppose $x \in (M^\perp)^\perp$. Since $H = M \oplus M^\perp$ we have $x = \mu + \nu$ with $\mu \in M$ and $\nu \in M^\perp$. $x \perp M^\perp$ $\therefore x \perp \nu$
$\therefore$ $0 = \langle x, \nu\rangle = \langle \mu, \nu\rangle + \langle \nu, \nu\rangle = 0 + \|\nu\|^2$
$\therefore \nu = 0$ $\therefore x = \mu \in M$ $\therefore$
Hence $(M^\perp)^\perp \subset M$