MAT 551 - Homework Set 1

1. Let V be the set of all sequences $\mathbf{x} := \{x_n\}_{n=1}^{\infty}$ such that $x_j = 0$ for all but finitely many indices j. V is a linear space in the obvious way and we define the inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=1}^{\infty} x_n y_n.$$

   Show that V is not a Hilbert space.

2. We say that a sequence $\{\mathbf{x}_n\}$ in a Hilbert space H converges weakly to $\mathbf{x}$ if for all $\mathbf{y} \in H$ we have

$$\lim_{n\to\infty} \langle \mathbf{x}_n, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle.$$

   Show that the sequence $\{\mathbf{x}_n\}$ converges (strongly) to $\mathbf{x}$ iff it converges weakly to $\mathbf{x}$ and

$$\lim_{n\to\infty} \|\mathbf{x}_n\| = \|\mathbf{x}\|.$$

3. Use the Gram-Schmidt process to find an orthonormal basis for the span of $1, x, x^2$ in $L^2[-1, 1]$.

4. Let M and N be two closed subspaces of a Hilbert space H such that $M \subset N$. Show that $N^{\perp} \subset M^{\perp}$. Show that $\left(M^{\perp}\right)^{\perp} = M$.