Properties of the Fourier Transform

We define the space of <u>rapidly</u> <u>decreasing</u> <u>functions</u>

$$\mathcal{S} := \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; \sup_x \left| x^\ell \left(\frac{d}{dx}\right)^k f(x) \right| < \infty \;\; \forall\, k, \ell \in \mathbb{Z}_+ \right\}$$

Here $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$

Note that if L is a linear ordinary differential operator with constant coefficients and $p(x)$ is a polynomial with constant coefficients then if $f \in \mathcal{S}$ we also have $p(x) L f(x) \in \mathcal{S}$.

NOTE: $\mathcal{D}(\mathbb{R}) \subset \mathcal{S} \subset L_2(\mathbb{R}) \therefore \mathcal{S}$ is dense in $L_2(\mathbb{R})$

<u>Theorem</u> $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a bijection whose inverse (on $\mathcal{S}$) is given by

$$\mathcal{F}^{-1}(F)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega$$

<u>Proof:</u> First we show that if $F = \mathcal{F}(f)$ then $\omega^k \left(\frac{d}{d\omega}\right)^\ell F(\omega)$ is bounded for all $k, \ell \in \mathbb{Z}_+$:

$$\omega^k \left(\frac{d}{d\omega}\right)^\ell F(\omega) = \omega^k \int_{-\infty}^{\infty} \left(\frac{d}{d\omega}\right)^\ell f(x) e^{i\omega x}\, dx$$

$$= \omega^k \int_{-\infty}^{\infty} (ix)^\ell f(x) e^{i\omega x}\, dx = \int_{-\infty}^{\infty} (ix)^\ell f(x) \left(\frac{1}{i}\frac{d}{dx}\right)^k e^{i\omega x}\, dx$$

Integrate by parts k times : $\displaystyle\int_{-\infty}^{\infty} \left[\left(i \frac{d}{dx}\right)^k (ix)^\ell f(x)\right] e^{i\omega x}\, dx$

The function in the square brackets is a member of $\mathcal{S}$ and hence is absolutely integrable. Therefore, denoting the function in the brackets by g:

$$\left|\omega^k\left(\frac{d}{d\omega}\right)^l F(\omega)\right| \le \int_{-\infty}^{\infty} |g(x)|\, dx < \infty$$

Similarly $\mathcal{F}^{-1}(F) \in \mathcal{S}$ if $F \in \mathcal{S}$. We saw $\mathcal{F}^{-1} \circ \mathcal{F}(f) = f$ and in a similar way $\mathcal{F} \circ \mathcal{F}^{-1}(F) = F \ \forall F \in \mathcal{S}$. Hence $\mathcal{F}$ and $\mathcal{F}^{-1}$ are both bijections

<u>Lemma</u> For any $f \in \mathcal{S}$ we have

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega = 2\pi \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

<u>Proof:</u> Since $|\hat{f}(\omega)|^2 \ge 0$ we have

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = \lim_{R\to\infty} \int_{-R}^{R} |\hat{f}(\omega)|^2 d\omega =$$

$$\lim_{R\to\infty} \int_{-R}^{R} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \int_{-\infty}^{\infty} \overline{f(y)}\, e^{-i\omega y} dy\, d\omega$$

Fubini's theorem can be used to justify any interchange of integrals in the above expression. Also, the Dominated Convergence theorem can be used ~~the~~ to interchange $\lim_{R\to\infty}$ with $\int_{-\infty}^{\infty} dx$

Hence

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} f(x) \lim_{R\to\infty} \int_{-\infty}^{\infty} \overline{f(y)} \int_{-R}^{R} e^{i\omega(x-y)} d\omega\, dy\, dx$$

$$= \int_{-\infty}^{\infty} f(x) \lim_{R\to\infty} 2\pi \int_{-\infty}^{\infty} \overline{f(y)} \frac{\sin R(x-y)}{\pi(x-y)}\, dy\, dx$$

Recalling $\left\{ \frac{\sin R(x-y)}{\pi(x-y)} \right\}$ is a delta net

$$= 2\pi \int_{-\infty}^{\infty} f(x) \overline{f(x)}\, dx.$$

<u>Definition</u> Let $\mathcal{U} := \frac{1}{\sqrt{2\pi}} \mathcal{F}$

Let $\| \ \|$ denote the $L_2(\mathbb{R})$ norm. The above lemma shows $\|\mathcal{U}f\| = \|f\|$ $\forall$ $f \in \mathcal{S} \subset L_2(\mathbb{R})$ and $\mathcal{U}$ is surjective. ~~Hence~~ onto $\mathcal{S}$ Hence $\mathcal{U}$ can be extended to a unitary operator on all of $L_2(\mathbb{R})$. Equivalently $\mathcal{F}$ and $\mathcal{F}^{-1}$ extend to bounded linear operators on $L_2(\mathbb{R})$ with
$\|\mathcal{F}\| = \sqrt{2\pi}$, $\|\mathcal{F}^{-1}\| = \frac{1}{\sqrt{2\pi}}$

$\mathcal{U}^{-1} = \mathcal{U}^*$ so $\mathcal{F}^{-1} = \frac{1}{2\pi} \mathcal{F}^*$ and we have

<u>Theorem</u> (Parseval's or Plancherel's identity)
For all $f, g \in L_2(\mathbb{R})$ we have

$$\int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{g}(\omega)}\, d\omega = 2\pi \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx$$

Recall $\mathcal{U} = \frac{1}{\sqrt{2\pi}} \mathcal{F}$ is a unitary operator on $L_2(\mathbb{R})$

$$\mathcal{U}^2 f = \check{f}$$

$$\therefore \quad \mathcal{U}^4 = \text{identity}$$

By the spectral mapping theorem $\sigma(\mathcal{U}) \subseteq \{1, -1, i, -i\}$. In fact $\sigma(\mathcal{U}) = \sigma_p(\mathcal{U}) = \{1, -1, i, -i\}$ and there exists a basis for $L_2(\mathbb{R})$ of the form

$$\varphi_n(x) = p_n(x) e^{-x^2/2} \qquad n = 0, 1, 2, \ldots$$

p_n has degree n (Hermite polynomial)

if n = even all terms in p_n are even degree
if n = odd " " " " " odd "

$$\mathcal{U} \varphi_n = i^n \varphi_n$$

In particular $\mathcal{U} e^{-x^2/2} = e^{-x^2/2}$

$$\therefore \quad \mathcal{F}(e^{-x^2/2}) = \sqrt{2\pi}\, e^{-\omega^2/2}$$

Now we have $\mathcal{F}$ defined on $L_1(\mathbb{R}) \cup L_2(\mathbb{R})$

More Properties

These properties hold for functions in $\mathcal{S}$ and then can be extended to functions in $L_p(\mathbb{R})$ or even to distributions (later)

1. $\mathcal{F}(f) = F$ then $\mathcal{F}(f')(\omega) = -i\omega F(\omega)$

2. $\overline{\mathcal{F}^{-1}(\bar{g})} = \frac{1}{2\pi}\mathcal{F}(g)$ or $\mathcal{F}^{-1}(G) = \frac{1}{2\pi}\overline{\mathcal{F}(\bar{G})}$

3. $\hat{\hat{f}} = 2\pi \check{f}$ where $\check{f}(x) := f(-x)$

$\tilde{\tilde{F}} = 2\pi \check{F}$ where $\sim$ denotes $\mathcal{F}^{-1}$

5. If $f_a(x) := f(x-a)$ then $\hat{f_a}(\omega) := e^{ia\omega}\hat{f}(\omega)$

6. If $g(x) := x f(x)$ then $\hat{g}(\omega) = \frac{1}{i}\frac{d}{d\omega}\hat{f}(\omega)$

7. If f and g are in $L_1(\mathbb{R})$ then their convolution $h = f * g$ is also in $L_1(\mathbb{R})$
Moreover
$$\hat{h}(\omega) = \hat{f}(\omega)\,\hat{g}(\omega)$$

We also have

8. $g(x) := f(ax+b) \Rightarrow \hat{g}(\omega) = \frac{1}{a} e^{-\frac{i\omega b}{a}} \hat{f}\left(\frac{\omega}{a}\right)$

Fourier transforms of tempered distributions

Note: $\mathcal{S}$ is metrizable. We can, in other words, put a metric on $\mathcal{S}$ such that $f_n \to f$ in the metric iff

$$x^l\left(\frac{d}{dx}\right)^k f_n \to x^l\left(\frac{d}{dx}\right)^k f$$

uniformly as $n \to \infty$

$\mathcal{D} \subset \mathcal{S}$ and if $\varphi_\alpha \to \varphi$ in $\mathcal{D}$ then $\varphi_\alpha \to \varphi$ in $\mathcal{S}$. Hence any continuous linear functional on $\mathcal{S}$ is a continuous linear functional on $\mathcal{D}$ (check the logic!)

$\therefore$ $\mathcal{S}' \subset \mathcal{D}'$

$\mathcal{S}'$ is the space of tempered distributions

<u>Example</u> Let $f \in L_1^{loc}(\mathbb{R})$ such that $|f(x)| \le C|x|^N$ for $|x| \ge R$. Here C and R are positive numbers and $N \in \mathbb{Z}_+$ then $\varphi \to \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx$ is a tempered distribution. (Exercise: prove this)

<u>Example</u> $\delta^{(k)}$ is a ~~dist~~ tempered distribution

$$\langle \delta^{(k)}, \varphi \rangle = (-1)^k \langle 1, \varphi^{(k)} \rangle = (-1)^k \int_{-\infty}^{\infty} \varphi^{(k)}(x)\,dx$$

and this extends to $\varphi \in \mathcal{S}$.

<u>Definition</u> Let σ be a tempered distribution then we define its Fourier transform $\hat{\sigma}$ by (extending Parseval's identity)

$$\langle \hat{\sigma}, \varphi \rangle := \langle \sigma, \hat{\varphi} \rangle$$

We see that if σ is a regular distribution defined by an $L_2(\mathbb{R})$ function $\sigma(x)$ then

$$\langle \hat{\sigma}, \varphi \rangle = \int_{-\infty}^{\infty} \sigma(x)\,\hat{\varphi}(x)\,dx = \int_{-\infty}^{\infty} \hat{\sigma}(x)\,\varphi(x)\,dx$$

by Parseval's identity. Hence this definition of Fourier transforms extends the definition of Fourier transforms on $L_2(\mathbb{R})$.

<u>Example</u> $\hat{\delta} = 1$ $\quad \hat{1} = 2\pi\delta$

To see this

$$\langle \hat{\delta}, \varphi \rangle = \langle \delta, \hat{\varphi} \rangle = \left. \int_{-\infty}^{\infty} e^{i\omega x}\varphi(x)\,dx \right|_{\omega=0}$$

$$= \int_{-\infty}^{\infty} 1 \cdot \varphi(x)\,dx = \langle 1, \varphi \rangle$$

$$\langle \hat{1}, \varphi \rangle = \langle 1, \hat{\varphi} \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x)\,e^{i\omega x}\,dx\,d\omega$$

$$= \lim_{R\to\infty} \int_{-R}^{R} \int_{-\infty}^{\infty} \varphi(x)\,e^{i\omega x}\,dx\,d\omega =$$

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \varphi(x) \int_{-R}^{R} e^{i\omega x}\,d\omega = \lim_{R\to\infty} 2\pi \int_{-\infty}^{\infty} \varphi(x)\,\frac{\sin Rx}{\pi x}\,dx$$

$$= 2\pi \varphi(0) = 2\pi \langle \delta, \varphi \rangle$$

Physicists often write the equation $\hat{1} = 2\pi\delta$ as

$$\int_{-\infty}^{\infty} e^{i\omega x}\, dx = 2\pi\, \delta(\omega)$$

where the integral on the left obviously does not exist. Now you know what that equation really means

Fourier series as distributions

Let f be a 2π-periodic function, $f \in C^k(\mathbb{R})$. If f is piecewise smooth and continuous then its Fourier series converges uniformly. If $f \in C^k(\mathbb{R})$ we can get the following estimate

$$c_m = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-imx}\, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} f(x)\left(i\frac{d}{dx}\right)^k e^{-imx}/m^k\, dx$$

$$= \frac{1}{2\pi}(-i)^k \int_0^{\pi} f^{(k)}(x)\, e^{-imx} dx \,/\, m^k$$

$$\leq \text{cst} / m^k$$

So if $k \geq 2$ then we have by the Weierstrass M-test that the Fourier series converges uniformly

Now suppose $f(x) = \sum_{m=-\infty}^{\infty} c_m e^{imx}$ where the series converges uniformly $\therefore f \in L_1^{loc}(\mathbb{R})$ and defines a distribution

$$\langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\, dx .$$ The distributional derivative

$$\langle f^{(\ell)}, \varphi \rangle = (-1)^\ell \langle f, \varphi^{(\ell)} \rangle$$

$$= (-1)^\ell \sum_{m=-\infty}^{\infty} c_m \langle e^{imx}, \varphi^{(\ell)} \rangle$$

$$= \sum_{m=-\infty}^{\infty} c_m \left\langle \left(\frac{d}{dx}\right)^\ell e^{imx}, \varphi \right\rangle$$

$$= \sum_{m=-\infty}^{\infty} \langle c_m (im)^\ell e^{imx}, \varphi \rangle$$

In this sense, the termwise differentiated series has meaning

$$f^{(\ell)} \sim \sum_{m=-\infty}^{\infty} c_m (im)^\ell e^{imx}$$

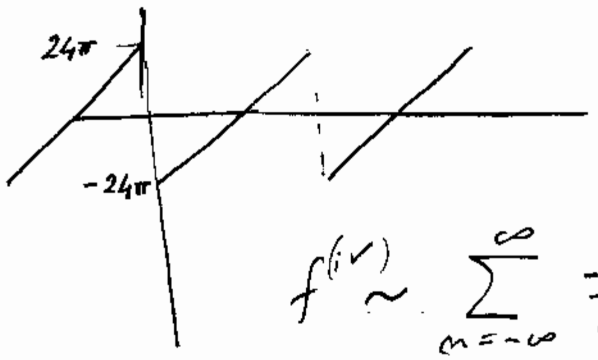
represents a distribution

Example Let $f(x) = x^4 - 4\pi x^3 + 4\pi^2 x^2 - \frac{16\pi^4}{30}$ $0 \le x \le 2\pi$ extended as a 2π-periodic function. This function can be seen to be in $C^3(\mathbb{R})$ and hence we know the coefficients are bounded by cst/m^3. In fact they are

for $m \neq 0$, $c_m = \frac{-24}{m^4}$, $c_0 = 0$

$f'''(x) = 24(x - \pi)$ $0 < x < 2\pi$

& 2π-periodic



$$\therefore f^{(iv)} = -48\pi \sum_{k=-\infty}^{\infty} \delta(x - 2k\pi)$$

$$f^{(iv)} \sim \sum_{m=-\infty}^{\infty} \frac{-24}{m^4} (im)^4 e^{imx}$$

Hence we have

$$\sum_{m=-\infty}^{\infty} \delta(x-2m\pi) = \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} e^{imx} \qquad (*)$$

Next, as a curiosity, let's see what happens if we take the Fourier transform of $(*)$. We know $\hat{\delta}=1$ and $\hat{f_a}(\omega) = e^{ia\omega}\hat{f}(\omega)$. Therefore

$\hat{\delta}_{2m\pi}(\omega) = e^{i2m\pi\omega}$. Let $E_m(\tau) = e^{im\tau}$

$$\langle \hat{E}_m, \varphi \rangle = \langle E_m, \hat{\varphi} \rangle = \langle e^{im\omega}, \int_{-\infty}^{\infty} e^{i\omega x}\varphi(x)\,dx\,d\omega \rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega(x-m)}\varphi(x)\,dx\,d\omega = \int_{-\infty}^{\infty} \hat{\varphi}_m\,d\omega$$

where $\varphi_m(\omega) = \varphi(\omega-m)$. The last integral can be interpreted as $2\pi \mathcal{F}^{-1}(\hat{\varphi}_m)\Big|_{x=0} = 2\pi\varphi(m)$

Hence $\hat{E}_m = 2\pi\delta_m$. Now we are ready to take the Fourier transform of $(*)$

$$\sum_{m=-\infty}^{\infty} e^{i2m\pi\omega} = \sum_{m=-\infty}^{\infty} \delta_m(\omega) \qquad (**)$$

In particular we get the formulas

$$\mathcal{F}\Big(\sum_{m=-\infty}^{\infty} \delta(x-2m\pi)\Big) = \sum_{m=-\infty} \delta(\omega-m)$$

$$\mathcal{F}\Big(\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{imx}\Big) = \sum_{m=-\infty}^{\infty} e^{i2m\pi\omega}$$