Hilbert Space Isomorphisms

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces with inner products $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$, and corresponding norms $\|\ \|_1$ and $\|\ \|_2$. If $L : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator then $L^* : \mathcal{H}_2 \to \mathcal{H}_1$ is the bounded linear operator that satisfies

$$\langle Lx, y\rangle_2 = \langle x, L^*y\rangle_1 \quad \forall x \in \mathcal{H}_1 \text{ and } \forall y \in \mathcal{H}_2.$$

An *isometry* from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a linear transformation L from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $\|Lx\|_2 = \|x\|_1$ for all $x \in \mathcal{H}_1$. We say that L is an *isometric isomorphism* if L is a bijective isometry. The following are equivalent:

1. $U : \mathcal{H}_1 \to \mathcal{H}_2$ is an isometric isomorphism.
2. $U : \mathcal{H}_1 \to \mathcal{H}_2$ is a surjective isometry.
3. $U : \mathcal{H}_1 \to \mathcal{H}_2$ is surjective and
$$\langle Ux, Uy\rangle_2 = \langle x, y\rangle_1 \quad \forall x, y \in \mathcal{H}_1.$$
4. $U^* : \mathcal{H}_2 \to \mathcal{H}_1$ is the inverse for $U : \mathcal{H}_1 \to \mathcal{H}_2$.

The implication $1 \Rightarrow 2$ is immediate. The implication $2 \Rightarrow 3$ follows from considering

$$\langle U(x + \alpha y), U(x + \alpha y)\rangle_2 = \langle x + \alpha y, x + \alpha y\rangle_1,$$

which leads to

$$\langle Ux, Ux\rangle_2 + \langle Ux, U\alpha y\rangle_2 + \langle U\alpha y, Ux\rangle_2 + \langle U\alpha y, U\alpha y\rangle_2 = \langle x, x\rangle_1 + \langle x, \alpha y\rangle_1 + \langle \alpha y, x\rangle_1 + \langle \alpha y, \alpha y\rangle_1,$$

and using the fact that U is isometric,

$$E(\alpha) := \langle Ux, \alpha Uy\rangle_2 + \langle \alpha Uy, Ux\rangle_2 = \langle x, \alpha y\rangle_1 + \langle \alpha y, x\rangle_1.$$

This can be written

$$E(\alpha) := 2\Re\langle Ux, \alpha Uy\rangle_2 = 2\Re\langle x, \alpha y\rangle_1,$$

where $\Re$ denotes “real part”. By considering both $E(1)$ and $E(i)$ we see that $\langle Ux, Uy\rangle_2$ and $\langle x, y\rangle_1$ have the same real part and the same imaginary part, i.e.

$$\langle Ux, Uy\rangle_2 = \langle x, y\rangle_1 \quad \forall x, y \in \mathcal{H}_1.$$

The proof of $3 \Rightarrow 4$ is fairly straight forward: $\forall x, y \in \mathcal{H}_1$

$$\langle x, y\rangle_1 - \langle Ux, Uy\rangle_2 = \langle x, y\rangle_1 - \langle U^*Ux, y\rangle_1 = \langle x - U^*Ux, y\rangle_1.$$

But this means that $U^*Ux = x$ for all $x \in \mathcal{H}_1$, i.e. $U^*U = I$. Hence $UU^*Ux = Ux$ for all $x \in \mathcal{H}_1$. But since U is surjective this implies that we also have $UU^* = I$. Therefore $U^* = U^{-1}$. The implication $4 \Rightarrow 1$ is easy. Since U has an inverse it must be a bijection. Moreover

$$\|Ux\|_2^2 = \langle Ux, Ux\rangle_2 = \langle x, U^*Ux\rangle_1 = \langle x, x\rangle_1 = \|x\|_1^2.$$