A Note on Nonseparable Hilbert Spaces

Let $\mathcal{H}$ be a nonseparable Hilbert space and let $\mathcal{B} := \{e_\alpha\}_{\alpha\in A}$ be a maximal orthonormal family in $\mathcal{H}$. Let x be any vector in the Hilbert space and for each positive integer j define

$$A_{xj} := \{\alpha \in A \mid |\langle x, e_\alpha\rangle| > \|x\|/\sqrt{j}\}.$$

We easily see that A_{xj} can have no more than j members. To see this, suppose that it had at least $j+1$ members, α_1, α_2, . . . , α_{j+1}. Let

$$S := \mathrm{Span}\{e_{\alpha_i} \mid i = 1, 2, \cdots, j+1\},$$

and let P_S be the orthogonal projection operator onto S. By Bessel's inequality

$$\|x\|^2 \geq \sum_{i=1}^{j+1} |\langle x, e_\alpha\rangle|^2 > \sum_{i=1}^{j+1} \frac{\|x\|^2}{j} > \|x\|^2,$$

which is a contradiction. Now let

$$A_x := \{\alpha \in A \mid |\langle x, e_\alpha\rangle| > 0\}.$$

This set is obviously the union of all the finite set A_{xj} and hence is a countable set. Relabeling we can say that

$$A_x = \{\alpha_1, \alpha_1, \cdots\},$$

and we define

$$\mathcal{H}_0 = \overline{\mathrm{Span}\{e_{\alpha_1}, e_{\alpha_2}, \cdots\}}$$

Let P_0 be the orthogonal projection operator onto $\mathcal{H}_0$. Then we have the unique decomposition

$$x = P_0 x = \sum_{i=1}^{\infty} \langle x, e_{\alpha_i}\rangle e_{\alpha_i}.$$

We therefore see that $\mathcal{B}$ is a basis for $\mathcal{H}$. Knowing this we see that all the usual statements that are equivalent to completeness of an orthonormal system $\mathcal{B}$ (maximality of $\mathcal{B}$, Parseval's identity, Bessel's equality, etc) are still equivalent in case the Hilbert space is not separable.