Let

$$f(\epsilon) = \int_{1/\epsilon}^{\infty} t^{-1}\mathrm{e}^{-t}\,dt.$$

We want to integrate by parts, however it will prove to be more efficient to first obtain

$$k!\int_{1/\epsilon}^{\infty} t^{-k-1}\mathrm{e}^{-t}\,dt = -k!t^{-k-1}\mathrm{e}^{-t}\Big|_{1/\epsilon}^{\infty} - (k+1)!\int_{1/\epsilon}^{\infty} t^{-k-2}\mathrm{e}^{-t}\,dt. \tag{1}$$

To see this we integrate by parts, letting $u = t^{-k-1}$ and $v' = \mathrm{e}^{-t}$, so that $u' = -(k+1)t^{-k-2}$ and $v' = -\mathrm{e}^{-t}$. The above formula simplifies to

$$k!\int_{1/\epsilon}^{\infty} t^{-k-1}\mathrm{e}^{-t}\,dt = k!\epsilon^{k+1}\mathrm{e}^{-1/\epsilon} - (k+1)!\int_{1/\epsilon}^{\infty} t^{-k-2}\mathrm{e}^{-t}\,dt.$$

Applying formula (1) to the definition of $f(\epsilon)$ we see that

$$f(\epsilon) = \epsilon\mathrm{e}^{-1/\epsilon} - \int_{1/\epsilon}^{\infty} t^{-2}\mathrm{e}^{-t}\,dt.$$

Repeating the process, we get

$$f(\epsilon) = (\epsilon - \epsilon^2)\mathrm{e}^{-1/\epsilon} + 2!\int_{1/\epsilon}^{\infty} t^{-3}\mathrm{e}^{-t}\,dt.$$

Inductively we obtain

$$f(\epsilon) = \mathrm{e}^{-1/\epsilon}\sum_{n=0}^{N-1}(-1)^n\epsilon^{n+1}\,n! + (-1)^N N!\int_{1/\epsilon}^{\infty} t^{-N-1}\mathrm{e}^{-t}\,dt.$$

So we define the series

$$f(\epsilon) \sim \mathrm{e}^{-1/\epsilon}\sum_{n=0}^{\infty}(-1)^n\epsilon^n\,n!. \tag{2}$$

To show that this is an asymptotic series we need to look at the limit of the ratio of the error of the partial sum divided by the last term of the partial sum. That is to say we must show that $\lim_{\epsilon\to 0} R(\epsilon) = 0$ where

$$R(\epsilon) := \frac{(-1)^N N!\int_{1/\epsilon}^{\infty} t^{-N-1}\mathrm{e}^{-t}\,dt}{\mathrm{e}^{-1/\epsilon}(-1)^{N-1}\epsilon^{N-1}\,(N-1)!}.$$

We apply formula (1) to the numerator:

$$R(\epsilon) = \lim_{\epsilon\to 0}\frac{N!\epsilon^{N+1}\mathrm{e}^{-1/\epsilon} - (N+1)!\int_{1/\epsilon}^{\infty} t^{-N-2}\mathrm{e}^{-t}\,dt}{-\mathrm{e}^{-1/\epsilon}\epsilon^N\,(N-1)!}.$$

Now we estimate

$$\int_{1/\epsilon}^{\infty} t^{-N-2}\mathrm{e}^{-t}\,dt \le \mathrm{e}^{-1/\epsilon}\int_{1/\epsilon}^{\infty} t^{-N-2}\,dt = \mathrm{e}^{-1/\epsilon}\epsilon^{N+1}/(N+1).$$

So

$$|R(\epsilon)| = \left|\frac{N!\epsilon^{N+1}\mathrm{e}^{-1/\epsilon} - (N+1)!\int_{1/\epsilon}^{\infty} t^{-N-2}\mathrm{e}^{-t}\,dt}{-\epsilon^N\,(N-1)!}\right| \le 2N(N+1)\epsilon \to 0 \text{ as } \epsilon \to 0+$$

It is easily seen, however, that the series (2) has radius of convergence 0.