Homework A

1. Consider the equation for the vibrating string with ends that are attached to elastic supports:

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L,$$

$$u_x(0,t) = \alpha u(0,t), \quad u_x(L,t) = -\alpha u(L,t),$$

where α is a positive number.

   (a) Separate variables and obtain the appropriate Sturm-Liouville problem.

   (b) Solve the Sturm-Liouville problem. Use a graph to explain the values of the eigenvalues. Assume that $2L\alpha/\pi$ is not an odd integer.

   (c) Find the solution for the PDE with initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

2. Consider the problem

$$u_t = k u_{xx}, \ 0 < x < L, \ t > 0,$$

with auxiliary conditions

$$u(x,0) = f(x), \quad u_x(0,t) = -\frac{\alpha}{L} u(0,t), \quad u_x(L,t) = \frac{\beta}{L} u(L,t),$$

with $0 < \alpha \le \beta$.

   (a) Show that the associated eigenvalue problem will have 2 negative eigenvalues, infinitely many positive eigenvalues and no zero eigenvalue if

$$\frac{1}{\alpha} + \frac{1}{\beta} < 1.$$

   Hint: When looking for the 2 negative eigenvalues you will come to an equation of the form

$$\tanh(z) = h(z), \quad z > 0.$$

   You will need to show that the graphs of $y = \tanh(z)$ and $y = h(z)$ cross twice. An "experiment" with a CAS such as Maple or MATLAB may help to give you an idea of how to do this.

   (b) Find the solution to the PDE assuming that $0 < \alpha \le \beta$ and $\frac{1}{\alpha} + \frac{1}{\beta} < 1$. Also, you may assume that $2\sqrt{\alpha\beta}/\pi$ is not an odd integer.