MAT 462 Final Exam

Fall 2006

Do 4 of the following problems

Try not to spend more than 25 minutes on each problem.

1. Solve the problem

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0$$
$$u(x,0) = f(x), \quad u_x(0,t) = 0, \quad u_x(1,t) = u(1,t).$$

2. Solve the problem

$$u_{tt} - u_{xx} = 1, \quad 0 < x < L,$$
$$u(0,t) = 3, \ u_x(L,t) = 5,$$
$$u(x,0) = 0, \ u_t(x,0) = 0.$$

3. Solve the problem

$$u_{tt} - c^2 u_{xx} = 0 \text{ on } 0 < x < \infty, \ t > 0$$
$$u_x(0,t) = \phi(t), \text{ for } t > 0; \quad u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) \text{ for } x > 0.$$

   Here c is a positive constant.

4. Suppose $\Delta u + f(r,\theta) = 0$ on the disk of radius 1 (i.e. in the region $r < 1$) with $u(1,\theta) \leq 0$, and suppose that M is a constant such that $f(r,\theta) \leq 4M$. Use the maximum principle to show that $u(r,\theta) \leq M(1-r^2)$. Note that $\Delta u = r^{-1}(ru_r)_r + r^{-2}u_{\theta\theta}$.

5. Use an energy integral to prove the uniqueness of bounded, finite energy solutions to the following Cauchy problem on $(-\infty, \infty)$ for $t > 0$:

$$u_t = u_{xx} + f(x,t),$$
$$u(x,0) = \phi(x),$$
$$u_x(x,t) \to 0 \text{ as } x \to \pm\infty.$$

6. Solve the Dirichlet problem :

$$\Delta u = 0, \quad 1 < r < e, \ 0 < \theta < \pi/2,$$
$$u(1,\theta) = 0, \quad u(e,\theta) = 0, \quad u(r,0) = 0, \quad u(r,\pi/2) = f(r).$$

   Hint: $r^{\pm i\omega} = e^{\pm i\omega \ln r} = \cos(\omega \ln r) \pm i \sin(\omega \ln r)$.