Supplementary Problems

The problems marked with ⓒ are challenge problems and are only appropriate for MAT371 students.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + x^5$ and let g denote the inverse function. Find $g'(2)$.

2. ⓒLet $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(0) = 0$, $f(x) = x^4 \sin(1/x)$ for $x > 0$. Define

$$F(x) := \int_0^{\sqrt{x}} f(t)\, dt, \quad x \geq 0.$$

   (a) Find $F'(x)$ for $x > 0$. *Ans.* $(1/2)x^{3/2} \sin(x^{-1/2})$.

   (b) Find $F'(0)$. *Ans.* 0, *using the definition of derivative!.*

   (c) Is F' continuous at 0? *Ans. Yes.*

   (d) Find $F''(x)$ for $x > 0$. *Ans.* $(3/4)x^{1/2} \sin(x^{-1/2}) - (1/4)\cos(x^{-1/2})$.

   (e) Find $F''(0)$. *Ans.* 0.

   (f) Is F'' continuous at 0? *Ans. No.*

3. Recall that $\sinh(x) := (e^x - e^{-x})/2$ and $\cosh(x) := (e^x + e^{-x})/2$. .

   (a) Show $\cosh(x)^2 = 1 + \sinh(x)^2$.

   (b) Show that

$$\frac{d}{dx} \sinh(x) = \cosh(x).$$

   (c) Let $f(x) := \sinh(x)$ and let g be the inverse function of f. Using the inverse function theorem (i.e. without computing g) obtain $g'(y)$ explicitly, i.e. a formula in y.

4. Use the change of variable theorem to evaluate:

   (a)

$$\int_0^3 \frac{x+1}{x^2 + 2x + 7}\, dx,$$

   (b)

$$\int_0^b \exp(-x^2)\, x\, dx,$$

   (c)

$$\int_a^b 3^{\cos(x)} \sin(x)\, dx.$$

5. Let f be a periodic function with period π, i.e. $f(x+\pi) = f(x) \quad \forall x \in \mathbb{R}$. Define

$$G(x) := \int_x^{x+\pi} f(t)\, dt.$$

Evaluate $G'(x)$ and show that $G(x)$ is independent of x

6. Evaluate:

   (a)
$$\lim_{x\to 0} \frac{e^x - \cos(x) - \sin(x)}{x^2}$$

   (b)
$$\lim_{n\to\infty} \sum_{i=1}^{n} i^2/n^3.$$

   (c) The upper sum, lower sum and Riemann sum for the function $f(x) = 1 + x - x^2$ on $[0, 2]$ if the partition is $\{0 < 1 < 1.5 < 2\}$ and the marking is $\{0, 1.1, 1.5\}$.

   (d)
$$\frac{d}{dx} \int_{x^2}^{\ln(x)} \tan(t)\, dt$$

7. ©Let f and g be two functions that are continuous on some neighborhood of x_0 and suppose $g(x_0 \neq 0$. Use the Mean Value Theorem for Integrals to prove that

$$\lim_{x\to x_0} \frac{\int_{x_0}^{x} f(t)\, dt}{\int_{x_0}^{x} g(t)\, dt} = f(x_0)/g(x_0).$$

8. ©Let $f(x, y)$ be a function of two variables defined on the plane. Let $g(x, y) := f_{xx}(x, y)$ be the second derivative of f with respect to x. Suppose that g is a bounded function (i.e. there is a positive number K such that $|g(x, y)| \leq K \quad \forall (x, y)$. Prove that

$$\frac{d}{dx} \int_a^b f(x, y)\, dy = \int_a^b f_x(x, y)\, dy,$$

where f_x is the first derivative of f with respect to x.

The last problem together with the Leibnitz rule from class can be used to derive the "full" Leibnitz rule:

$$\frac{d}{dt} \int_{\alpha(t)}^{\beta(t)} f(\gamma(t), y)\, dy = f(\gamma(t), \beta(t))\beta'(t) - f(\gamma(t), \alpha(t))\alpha'(t) + \gamma'(t) \int_{\alpha(t)}^{\beta(t)} f_x(\gamma(t), y) dy.$$