Sequences and Accumulation Points

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, that is to say, a function

$$a : \mathbb{N} \to \mathbb{R}, \quad a(n) = a_n,$$

and let the range, $a(\mathbb{N})$ be denoted by $\mathcal{R}$.

First, an important theorem:

Theorem. Every bounded sequence has a convergent subsequence.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence. If the range is finite then there is a value L in the range such that $a_n = L$ for infinitely many indices $n_1, n_2, \cdots$. This yields a convergent subsequence $\{a_{n_i}\}_{i=1}^{\infty}$. If the range is infinite, then by the Bolzano-Weierstrass theorem it has an accumulation point L. Now we can inductively find a subsequence as follows. Choose $a_{n_1} = a_1$, and for each $i > 1$ choose a_{n_i} such that $n_i > n_{i-1}$ and also $|a_{n_i} - L| < 1/i$. This subsequence evidently converges to L.

	$\lim_{n\to\infty} a_n = L$	$\mathcal{R}$ is bounded	$\mathcal{R}$ is unbounded
$\mathcal{R}$ is finite $\lvert\mathcal{R}\rvert < \aleph_0$	$L \in \mathcal{R}$ $\exists N \in \mathbb{N}$ s.t. $a_n = L \quad \forall n \geq N$ Example 1	$\exists$ convergent subsequence Example 2	Impossible Finite sets are bounded
$\mathcal{R}$ is infinite $\lvert\mathcal{R}\rvert = \aleph_0$	L is an acc. point of $\mathcal{R}$ Example 3	$\mathcal{R}$ has acc. points If P is an acc. pt. of $\mathcal{R}$ $\forall$ acc. pt. P of $\mathcal{R}$ $\exists$ subseq. convergent to P Example 4	Various possibilities Examples 5, 6

Example 1. $\{1, 2, 3, 2, 3, 1, 3, 3, 3, 3, \text{ etc }\}$, $a_n = 3 \quad \forall n \geq 7$.

Example 2. $a_n = (-1)^n$ with subsequences $\{1, 1, 1, \cdots\}$ and $\{-1, -1, -1, \cdots\}$.

Example 3. $\{1/n\}_{n=1}^{\infty}$.

Example 4. $\{(-1)^n + 1/n\}_{n=1}^{\infty}$ with subsequences $\{1 + 1/(2k)\}_{k=1}^{\infty}$ and $\{-1 + 1/(2k-1)\}_{k=1}^{\infty}$.

Example 5. $\{1, 2, 3, \cdots\}$ has no accumulation points.

Example 6. $\{1, 1, 2, 1/2, 3, 1/3, 4, 1/4, \cdots\}$ has 0 as an accumulation point.