Derived Set, Closure, Interior, and Boundary

We have the following definitions:

- Let A be a set of real numbers. We use $\mathbf{d}(A)$ to denote the *derived set* of A, that is the set of all accumulation points of A. This set is sometimes denoted by A'.
- The *closure* of A is the set $\mathbf{c}(A) := A \cup \mathbf{d}(A)$. This set is sometimes denoted by $\overline{A}$.
- The complement of A is the set $\mathcal{C}(A) := \mathbb{R} \setminus A$. The complement of A is sometimes denoted by A^c.
- The *interior* of A, denoted $\mathbf{i}(A)$, is the set consisting of all points that lie in A together with a neighborhood. That is $x \in \mathbf{i}(A)$ iff for some positive number ϵ we have $(x - \epsilon, x + \epsilon) \subset A$. The interior of A is sometimes denoted by A^o.
- The *boundary* of A is the set $\mathbf{b}(A) := \mathbf{c}(A) \setminus \mathbf{i}(A)$. The boundary of A is sometimes denoted by ∂A.

Note that $\mathbf{b}$, $\mathbf{c}$, $\mathbf{i}$, $\mathbf{d}$, and $\mathcal{C}$ are all functions from the power set of $\mathbb{R}$ to itself.
We have the following results:

1. $\mathbf{c}(A)$ is a closed set. It is the smallest closed set containing A.
2. $\mathbf{i}(A)$ is an open set. It is the largest open set contained by A.
3. $\mathbf{i}(A) = \mathcal{C} \circ \mathbf{c} \circ \mathcal{C}(A)$.
4. $\mathbf{c}(A) = \mathcal{C} \circ \mathbf{i} \circ \mathcal{C}(A)$.
5. $\mathbf{b}(A) = \mathcal{C}[\mathbf{i}(A) \cup \mathbf{i}(\mathcal{C}(A))]$. So $\mathbf{b}(A) = \mathbf{b}(\mathcal{C}(A))$.
6. $\mathbf{b}(A) = [\mathbf{c}(A) \setminus A] \cup [\mathbf{c}(\mathcal{C}(A)) \setminus \mathcal{C}(A)]$.

Example Let $A = [\mathbb{Q} \cap (0, \infty)] \cup \{-1\} \cup (-3, -2]$. Find all the corresponding sets that appear above:
$\mathbf{d}(A)$
$\mathbf{c}(A)$
$\mathcal{C}(A)$
$\mathbf{i}(A)$
$\mathbf{b}(A)$
$\mathcal{C} \circ \mathbf{c} \circ \mathcal{C}(A)$
$\mathcal{C} \circ \mathbf{i} \circ \mathcal{C}(A)$
$\mathcal{C}[\mathbf{i}(A) \cup \mathbf{i}(\mathcal{C}(A))]$
$[\mathbf{c}(A) \setminus A] \cup [\mathbf{c}(\mathcal{C}(A)) \setminus \mathcal{C}(A)]$.
Suggestion: use $\mathbb{I}$ to denote the irrational numbers.

Exercises. Prove the above results 1 - 6.