The Riemann integral

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. By a **partition** we mean a set of points

$$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b.$$

These points are called **mesh points**. We will denote this partition by P. A partition Q is called a **refinement** of P if $P \subset Q$. The graph of the the function f lies in a vertical strip in the xy-plane: $\{(x, y) \ : \ a \leq x \leq b\}$. This strip consist of N **panels** $\{(x, y) \ : \ x_{i-1} \leq x \leq x_i\}$. Loosely speaking, on this panel the graph of f varies between a minimum height m_i and a maximum height M_i. More precisely

$$m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}, \quad M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}.$$

We can now define the **lower Riemann sum** $L(P, f)$ and the **upper Riemann sum** $U(P, f)$:

$$L(P, f) := \sum_{i=1}^{N} m_i \Delta x_i, \quad U(P, f) := \sum_{i=1}^{N} M_i \Delta x_i.$$

Note that

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a) \tag{1}$$

Suppose we refine the partition P by adjoining one more mesh point z. Suppose j is the integer such that $x_{j-1} \leq z \leq x_j$ and let $\tilde{P}$ denote the refined partition: $\tilde{P} = P \cup \{z\}$. The extra point z divides the j^{th} panel of the original partition into a left panel and a right panel. Let

$$m_i^L = \inf\{f(x) \mid x_{i-1} \leq x \leq z\}, \quad M_i^L = \sup\{f(x) \mid x_{i-1} \leq x \leq z\},$$

$$m_i^R = \inf\{f(x) \mid z \leq x \leq x_i\}, \quad M_i^R = \sup\{f(x) \mid z \leq x \leq x_i\}.$$

When we compute the upper and lower Riemann sums for the refined partition we see that the only difference is that the j^{th} term is replaced by two new terms corresponding to the left and right panels:

$$m_i \Delta x_j \quad \text{becomes} \quad m_j^L(z - x_{j-1}) + m_j^R(x_j - z),$$

$$M_i \Delta x_j \quad \text{becomes} \quad M_j^L(z - x_{j-1}) + M_j^R(x_j - z).$$

Hence

$$L(P, f) - L(\tilde{P}, f) = (m_j - m_j^L)(z - x_{j-1}) + (m_j - m_j^R)(x_j - z), \tag{2}$$

$$U(P, f) - U(\tilde{P}, f) = (M_j - M_j^L)(z - x_{j-1}) + (M_j - M_j^R)(x_j - z). \tag{3}$$

Note that $m_j \leq m_j^L \leq M_j^L \leq M_j$ and $m_j \leq m_j^R \leq M_j^R \leq M_j$. This implies

$$L(P, f) \leq L(\tilde{P}, f) \leq U(\tilde{P}, f) \leq U(P, f).$$

Since we can think of a refinement Q of a partition P as being obtained by successively adding one point at a time we have:

Lemma 1. Let Q be a refinement of P then

$$L(P,f) \le L(Q,f) \le U(Q,f) \le U(P,f). \tag{4}$$

Note that equation (1) is actually a special case of equation (4).

Lemma 2. Let P_1 and P_2 be any two partitions of $[a,b]$ then $L(P_1,f) \le U(P_2,f)$ and hence

$$\sup_P L(P,f) \le \inf_P U(P,f).$$

Proof. Let $Q := P_1 \cup P_2$, then it refines both P_1 and P_2. Therefore, by lemma 1:

$$L(P_1,f) \le L(Q,f) \le U(Q,f) \le U(P_2,f).$$

We define $\Delta x_i := x_i - x_{i-1}$ and

$$\mu(P) := \max_i \Delta x_i.$$

The number $\mu(P)$ is called the mesh size of the partition P.

For later use we will also need another estimate:

Lemma 3. Let Q be a refinement of P obtained by adjoining K points, then

$$|L(P,f) - L(Q,f)| \le K(M-m)\mu(P), \quad |U(P,f) - U(Q,f)| \le K(M-m)\mu(P).$$

The proof is simply a consequence of the fact that neither suprema nor infima over any subinterval can differ by more than $M-m$. If we add an extra mesh point between x_{i-1} and x_i then the contribution from subinterval $[x_{i-1}, x_i]$ is not changed by more than $(M_i - m_i)\Delta x_i \le (M-m)\mu(P)$. So if we adjoin K points the change in either the lower sum or the upper sum is no more in magnitude than $K(M-m)\mu(P)$.

We define the **lower Riemann integral** $\underline{\int_a^b} f(x)\,dx$ and the **upper Riemann integral** $\overline{\int_a^b} f(x)\,dx$ as follows

$$\underline{\int_a^b} f(x)\,dx = \sup_P\ L(P,f), \text{ and } \quad \overline{\int_a^b} f(x)\,dx = \inf_P\ U(P,f).$$

By lemma 2 the lower Riemann integral is less than or equal to the upper Riemann integral. We say that the function f is Riemann integrable on $[a,b]$ if its lower and upper Riemann integrals have the same value. In that case we denote that common value by $\int_a^b f\,dx$, called the **Riemann integral** of f on $[a,b]$. We have the following important result:

Riemann Lemma. $f : [a,b] \to \mathbb{R}$ is Riemann integrable iff for any $\epsilon > 0$ there exists a partition Q such that $U(Q,f) - L(Q,f) < \epsilon$.

Proof. Suppose that f is Riemann integrable, then by the definitions of lower and upper Riemann integrals (which now have the same value) there exist partitions P_1 and P_2 such that

$$L(P_1,f) > \int_a^b f(x)\,dx - \epsilon/2, \quad U(P_1,f) < \int_a^b f(x)\,dx + \epsilon/2.$$

Let $Q = P_1 \cup P_2$, then

$$U(Q,f) - L(Q,f) \le U(P_2,f) - L(P_1,f) \le \left(U(P_2,f) - \int_a^b f(x)\,dx\right) + \left(\int_a^b f(x)\,dx - L(P_1,f)\right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conversely, if for each $\epsilon > 0$ we can find a partition Q such that $U(Q,f) - L(Q,f) < \epsilon$ then

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le U(Q,f) - L(Q,f) < \epsilon.$$

Since ϵ is an arbitrarily small positive number, the upper and lower integrals must have the same value.

Notation. We use $C[a, b]$ to denote the set of all continuous functions from $[a, b]$ to $\mathbb{R}$ and $R[a, b]$ to denote the set of all functions that are Riemann integrable on $[a, b]$.

Theorem. $C[a, b] \subset R[a, b]$.

Proof. Let $f \in C[a, b]$. Then f must, in fact, be uniformly continuous. Given any $\epsilon > 0$ we can find a $\delta > 0$ such that if $u, v \in [a, b]$ with $|u - v| < \delta$ then $|f(u) - f(v)| < \epsilon/(b - a)$. Let P be any partition with mesh size $\mu(P) < \delta$. This means that on each panel $M_i - m_i < \epsilon/(b - a)$. Therefore

$$U(P, f) - L(P, f) = \sum_i (M_i - m_i)\Delta x_i < \sum_i [\epsilon/(b - a)]\Delta x_i = \epsilon.$$

Integrability now follows from the Riemann lemma.

Telescoping sum. Note that

$$\sum_{j=1}^{N} [\gamma_j - \gamma_{j-1}] = \gamma_N - \gamma_0.$$

Such a sum is called a *telescoping sum*.

Theorem. Let $f : [a, b] \to \mathbb{R}$ be a monotone function. Then $f \in R[a, b]$.

Proof. We prove the case where f is increasing; the case where f is decreasing is handled similarly. Given any ϵ choose a positive integer N such that $(f(b) - f(a))(b - a)/N < \epsilon$. Choose the partition P with $x_i := a + i(b - a)/N$ so that $\Delta x_i = \Delta x := (b - a)/N$. From the fact that f is increasing it follows that $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Hence

$$U(P, f) - L(P, f) = \sum_{i=1}^{N} (f(x_i) - f(x_{i-1}))\Delta x$$

which is a telescoping sum that is equal to $(f(x_N) - f(x_0))\Delta x = (f(b) - f(a))(b - a)/N < \epsilon$. Integrability, once again, follows from the Riemann lemma.