Riemann sums

Let $P := \{x_0, x_1, \cdots, x_N\}$ be a partition. A set of points $\{t_1, t_2, \cdots, t_N\}$ is called a **marking** of the partition P if for each i we have $x_{i-1} \le t_i \le x_i$. We denote by P^T the partition P together with the marking T. We call P^T a **marked partition**. Sometimes we will simplify the notation and denote P^T by Π and define $\mu(\Pi) := \mu(P)$. Given such a marked partition we define the corresponding **Riemann sum** as

$$S(\Pi, f) = S(P^T, f) := \sum_{i=1}^{N} f(t_i)\, \Delta x_i.$$

Clearly, since $m_i \le f(t_i) \le M_i$, we have

$$L(P, f) \le S(P^T, f) \le U(P, f).$$

Theorem. Let $\{P_k \mid k \in \mathbb{N}\}$ be a family of partitions for $[a, b]$ such that

$$\lim_{k\to\infty} \mu(P_k) = 0.$$

For each k let T_k be a marking for P_k. If f is Riemann integrable on $[a, b]$ then

$$\lim_{k\to\infty} S(P_k^{T_k}, f) = \int_a^b f(x)\, dt.$$

Proof. Suppose that $m \le f(x) \le M$ for all $x \in [a, b]$. We assume $M > m$; the case $M = m$ would make the proof trivial. By the Riemann lemma, given any $\epsilon > 0$ we can find a partition Q such that $U(Q, f) - L(Q, f) < \epsilon/2$. Suppose that Q consists of K mesh points. Let k_0 be an integer such that

$$\mu(P_k) < \frac{\epsilon}{4K(M - m)} \quad \forall k \ge k_0.$$

Let $\tilde{P}_k$ be the the partition of P_k obtained by adjoining all the mesh points of Q. By our earlier lemma

$$|L(\tilde{P}_k, f) - L(P_k, f)| \le K(M - m)\mu(P_k), \quad |U(\tilde{P}_k, f) - U(P_k, f)| \le K(M - m)\mu(P_k).$$

Moreover, since

$$L(P_k, f) \le \int_a^b f(x)\, dx \le U(P_k, f), \quad , \quad L(P_k, f) \le S(P_k^{T_k}, f) \le U(P_k, f)$$

we have for $k \ge k_0$:

$$\left| \int_a^b f(x)\, dx - S(P_k^{T_k}, f) \right| \le U(P_k, f) - L(P_k, f)$$

$$\le U(\tilde{P}_k, f) - L(\tilde{P}_k, f) + 2K(M - m)\mu(P_k) \le U(Q, f) - L(Q, f) + \epsilon/2 < \epsilon.$$

We also have a converse to this theorem:

Theorem. Suppose $f : [a, b] \to \mathbb{R}$ is bounded and has the property that for any family of marked partitions $\{\Pi_k\}$ with $\mu(\Pi_k) \to 0$ as $k \to \infty$ the sequence $\{S(\Pi_k, f)\}_{k=1}^{\infty}$ converges. Then $f \in R[a, b]$, i.e. f is Riemann integrable on $[a, b]$.

We omit the proof but the idea is simple. We can construct partitions P_k such that $\mu(P_k) \to 0$ as $k \to \infty$. Then we can mark these partitions so that of $S(\Pi_{2k-1}, f) - L(P_k, f) < 1/k$ and $U(P_k, f) - S(\Pi_{2k}, f) < 1/k$. This implies that $U(P_k, f) - L(P_k, f)$ will converge to zero and therefore, by the Riemann lemma $f \in R[a, b]$

The algebra of integrable functions

Riemann sums are real handy to use to prove various algebraic properties for the Riemann integral. Here we list the main properties as a theorem and indicate how the proofs go.

Theorem. Let $f, g \in R[a,b]$ and let $k \in \mathbb{R}$. Then

1. $f+g \in R[a,b]$ and $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.
2. $kf \in R[a,b]$ and $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$.
3. If $f(x) \le g(x)\ \forall x \in [a,b]$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.
4. If $[p,q] \subset [a,b]$ then $f \in R[p,q]$.
5. If $a < c < b$ then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.
6. If $f([a,b]) \subset [c,d]$ and $\phi \in C[c,d]$ then $\phi \circ f \in R[a,b]$.
7. $|f| \in R[a,b]$ and $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.
8. $fg \in R[a,b]$.

Proof. Let $\Pi_k := P_k^{T_k},\ k = 1, 2, \cdots$ be marked partitions such that $\mu(\Pi_k) \to 0$ as $k \to \infty$. Then

$$R(\Pi_k, f+g) = R(\Pi_k, f) + R(\Pi_k, g).$$

the right hand side converges to $\int_a^b f(x)\,dx + \int_a^b fgx)\,dx$ as $k \to \infty$. Therefore $f+g$ is integrable and its integral is the sum of the integrals of f and g. The proofs of (2) and (3) follow similarly. To prove (4) we modify the partitions so that p and q are always among the mesh points. Let $U_*(P_k, f)$ and $L_*(P_k, f)$ be the sum of only those terms in the upper and lower Riemann sums that correspond to panels that lie between p and q. Then

$$0 \le U_*(P_k, f) - L_*(P_k, f) \le U(P_k, f) - L(P_k, f) \to 0,$$

and hence f is integrable on $[p,q]$. To prove (5) we arrange it so that $c \in P_k$ for all k. That is $c = x_{m(k)} \in P_k$. Then

$$R(\Pi_k, f) = R_L(\Pi_k, f) + R_R(\Pi_k, f),$$

where R_L denotes the sum of terms $1, 2, \cdot, m(k)$ and R_R denotes the sum of the remaining terms. Clearly

$$R_L(\Pi_k, f) \to \int_a^c f(x)\,dx, \quad R_R(\Pi_k, f) \to \int_c^b f(x)\,dx, \quad R(\Pi_k, f) \to \int_a^b f(x)\,dx.$$

The proof of (6) is somewhat lengthy and so we will do that last The first part of (7) follows immediately from (6) with $\phi(s) := |s|$. The second part follows from the triangle inequality applied to the Riemann sums:

$$|R(\Pi_k, f)| \le R(\Pi_k, |f|).$$

Taking limits we have the inequality in (7). We can also use (6) with $\phi(s) := s^2$ to deduce that f^2, g^2 and $(f+g)^2$ are all in $R[a,b]$. Therefore

$$fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right] \in R[a,b].$$

To prove (6), let $\epsilon > 0$ be given. Since ϕ is uniformly continuous and bounded, there exists a $\delta > 0$ such that $|\phi(u) - \phi(v)| < \epsilon/[2(b-a)]$ whenever $u, v \in [c,d]$ and $|u - v| < \delta$, and there exist numbers p and q such that $\phi([c,d]) \subset [p,q]$. Since f is integrable we can find a partition $P := \{x_0, x_1, \cdots, x_N\}$ such that

$$U(P, f) - L(P, f) = \sum_{i=1}^{N} (M_i - m_i)\Delta x_i < \eta := \epsilon\delta/[2(q-p)],$$

where M_i (resp. m_i) is the supremum (resp. infimum) of f on $[x_{i-1}, x_i]$. Let

$$J_\delta := \{j \in \mathbb{N} \, : 1 \leq j \leq N, \ M_j - m_j < \delta\}, \quad K_\delta := \{j \in \mathbb{N} \, : \, 1 \leq j \leq N, \ M_j - m_j \geq \delta\}.$$

We note that

$$\eta > \sum_{i=1}^{N} (M_i - m_i)\Delta x_i \geq \sum_{i \in K_\delta} \delta \Delta x_i.$$

This implies that

$$\sum_{i \in K_\delta} \Delta x_i < \eta/\delta = \epsilon/[2(q-p)].$$

Let M_i^* (resp. m_i^*) be the supremum (resp. infimum) of $\phi \circ f$ on $[x_{i-1}, x_i]$, then

$$U(P, \phi \circ f) - L(P, \phi \circ f) = \sum_{i=1}^{N} (M_i^* - m_i^*)\Delta x_i = \sum_{i \in J_\delta} (M_i^* - m_i^*)\Delta x_i + \sum_{i \in K_\delta} (M_i^* - m_i^*)\Delta x_i <$$

$$\sum_{i \in J_\delta} \epsilon \Delta x_i / [2(b-a)] + \sum_{i \in K_\delta} (q-p)\Delta x_i \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Warning. If $|f|$ is Riemann integrable it does not follow that f is Riemann integrable. Counterexample : $f(x) = 0$ if x rational, $f(x) = -1$ if x is irrational.

Remark. In general composition of two Riemann integrable functions is not integrable as can be seen from the following example. Let

$$f(x) = 0 \quad \forall x > 0 \text{ and } f(x) = 1 \quad \forall x \leq 0,$$

$$\psi(x) = 0 \quad \forall x \notin \mathbb{Q}, \quad \psi(p/q) = 1/q \quad \forall p \in \mathbb{Z}, \ \forall q \in \mathbb{N},$$

where we assume that p and q have no common factors in $\mathbb{N}$ except 1. The function ψ is continuous at all irrationals. It is a fact (not proven here) that a bounded function that is continuous on $[a, b]$ except at countably many points is Riemann integrable, and hence ψ is Riemann integrable on any bounded interval. The function f is integrable on $[0, 1]$, but the function $f \circ \psi$ is the Dirichlet function ($f(x)$ is zero on all rationals and 1 on all irrationals). The Dirichlet function is **not** Riemann integrable on any interval $[a, b]$ with $a < b$, since all upper sums have the value $b - a$ while all lower sums are zero. This provides an example of a composition of two Riemann integrable functions that is not Riemann integrable. There are even examples where f is Riemann integrable, ψ is continuous, but $f \circ \psi$ fails to be Riemann integrable.

Definition. Suppose that $\alpha < \beta$ the we define

$$\int_\beta^\alpha f(x)\, dx := -\int_\alpha^\beta f(x)\, dx.$$

Theorem. Let $f \in R[a, b]$ and $p, q, r \in [a, b]$ then

$$\int_p^q f(x)\, dx + \int_q^r f(x)\, dx = \int_p^r f(x)\, dx$$

irrespective of the relative sizes of p, q, and r.

The Fundamental Theorem of Calculus

Theorem. Suppose $f : [a, b] \to \mathbb{R}$ is a differentiable function and suppose f' is Riemann integrable on $[a, b]$. Then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Proof. Let $P_n,\ n = 1, 2, \cdots$ be partitions of $[a, b]$ such that $\lim_{n\to\infty} \mu(P_n) = 0$. That is $P_n = \{x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \cdots\}$. Let $T_n := \{t_0^{(n)}, t_1^{(n)}, t_2^{(n)}, \cdots\}$ be a marking of the partition P_n in such a way that

$$f'(t_j^{(n)}) = [f(x_j^{(n)}) - f(x_{j-1}^{(n)})]/[x_j^{(n)} - x_{j-1}^{(n)}].$$

This is possible by the Mean Value Theorem. Let Π_n be the partition P_n together with the marking T_n. Then

$$R(\Pi_n, f') = \sum_i f'(t_i^{(n)})\Delta x_i = \sum_i [f(x_i^{(n)}) - f(x_{i-1}^{(n)})] = f(b) - f(a). \tag{1}$$

The last equality follows from the fact that the sum is a telescoping series. Now, letting n tend to infinity we obtain the result we wanted.

Notation.

$$f(x)|_a^b := f(b) - f(a).$$

The formula for the Fundamental Theorem of Calculus then reads

$$\int_a^b f'(x)\,dx = f(x)|_a^b.$$

Leibniz's Rule

Lemma. Let $g : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and suppose that g is continuous at $c \in [a, b]$. Let

$$G(x) := \int_a^x g(s)\,ds.$$

Then G is differentiable at c and $G'(c) = g(c)$.

Proof. It suffices to show that

$$\lim_{t\to 0} \{[G(c + t) - G(c)]/t - g(c)\} = 0.$$

But the left side can be written as

$$\lim_{t\to 0} \frac{1}{t} \int_c^{c+t} [g(s) - g(c)]\,ds.$$

Given any $\epsilon > 0$ we can find a $\delta > 0$ such that $|g(s) - g(c)| < \epsilon$ whenever $|s - c| < \delta$. Hence, if $|t| < \delta$ then

$$\left| \frac{1}{t} \int_c^{c+t} [g(s) - g(c)]\,ds \right| < \epsilon.$$

A more general result is

Leibniz's Rule Let $g : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and suppose that g is continuous on $[a, b]$. Let $\alpha : [c, d] \to [a, b]$, and $\beta : [c, d] \to [a, b]$ be continuous functions that are differentiable at $x_0 \in [c, d]$. Let

$$\Phi(x) := \int_{\alpha(x)}^{\beta(x)} g(s)\, ds.$$

Then Φ is differentiable at x_0 and

$$\Phi'(x_0) = g(\beta(x_0))\beta'(x_0) - g(\alpha(x_0))\alpha'(x_0).$$

Proof [1] Choose $x_* \neq x_0$ and define

$$G(s) := \int_{x_*}^{s} g(t)\, dt,$$

then we can write

$$\Phi(x) = G(\beta(x)) - G(\alpha(x)).$$

The result then follows from the chain rule together with the above lemma.

A few more theorems on integrals.

Mean Value Theorem for Integrals Let $f \in C[a, b]$ and $g \in R[a, b]$ with $g(x) \geq 0 \quad \forall x \in [a, b]$. Then there exists a $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)\, dx = f(c) \int_a^b g(x)\, dx. \tag{2}$$

Proof. Let $m := \min(f)$, and $M := \max(f)$ and $G := \int_a^b g(x)\, dx$. Then since $g(x) \geq 0$ we have $mg(x) \leq f(x)g(x) \leq Mg(x)$ so that

$$mG = m \int_a^b g(x)\, dx \leq \int_a^b f(x)g(x)\, dx \leq M \int_a^b g(x)\, dx = MG. \tag{3}$$

If $G = 0$ then all integrals in (3) are zero and obviously (2) will be true for any choice of c. If $G > 0$ then $m \leq G^{-1} \int_a^b f(x)g(x)\, dx \leq M$ so that by the intermediate value theorem there exists a c such that $f(c) = G^{-1} \int_a^b f(x)g(x)$ for some $c \in [a, b]$.

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[1] Actually Leibniz's rule is a bit more general. Under the right hypotheses it is true that

$$\frac{d}{dx} \int_{\alpha(x)}^{\beta(x)} g(x, s)\, ds = \int_{\alpha(x)}^{\beta(x)} \frac{\partial g(x, s)}{\partial x}\, ds + g(x, \beta(x))\beta'(x) - g(x, \alpha(x))\alpha'(x).$$

Change of Variables Theorem. Let J be an interval, $\phi : [a, b] \to J$ a differentiable function with $\phi' \in R[a, b]$. Let $f : J \to \mathbb{R}$ be continuous. Then

$$\int_a^b f(\phi(t))\, \phi'(t)\, dt = \int_{\phi(a)}^{\phi(b)} f(x)\, dx.$$

Proof. Let

$$F(x) := \int_a^x f(\phi(t))\, \phi'(t)\, dt - \int_{\phi(a)}^{\phi(x)} f(s)\, ds.$$

Clearly $F(a) = 0$ and using Leibniz's rule we see that $F'(x) = 0$. This means $F(x) = 0 \quad \forall x \in [a, b]$. In particular $F(b) = 0$, which is precisely what the theorem asserts.

Integration by parts. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be differentiable functions with $f, g \in R[a, b]$. Then

$$\int_a^b f(x)g'(x)\, dx = -\int_a^b f'(x)g(x)\, dx + [f(b)g(b) - f(a)g(a)].$$

The proof is an immediate consequence of applying the Fundamental Theorem of Calculus to the function $f(x)g(x)$.

Taylor's Theorem. Let J be an interval, $a \in J$ and $f \in C^{n+1}(J)$, i.e. f is $n + 1$ times continuously differentiable on J. Then for all $x \in J$ we have

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + R_{n+1}(x), \tag{4}$$

where

$$R_{n+1}(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x - t)^n\, dt.$$

Moreover, for each $x \in J$ there is a number c between x and a such that

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1}.$$

Note. For $n = 0$ equation (2) reduces to the Fundamental Theorem of Calculus:

$$f(x) = f(a) + \int_a^x f'(t)\, dt.$$

The proof of the theorem for arbitrary n is effected by repeated integration by parts on the integral or, more simply, by mathematical induction.

The logarithm, exponential and power functions.

Definition We define the natural logarithm function as

$$\ln(x) := \int_1^x \frac{1}{t}\, dt.$$

Using this definition we can can derive all the properties of the natural logarithm

Theorem Let $a > 0$, $b > 0$.

1. $\ln : (0, \infty) \to (-\infty, \infty)$ is a bijection. It is strictly increasing: if $a > b$ then $\ln(a) > \ln(b)$.
2. $$\ln(ab) = \ln(a) + \ln(b).$$
3. If $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ then $$\ln(a^{m/n}) = \frac{m}{n} \ln(a).$$
4. $$\ln(1) = 0, \quad \ln(a/b) = \ln(a) - \ln(b).$$
5. $$[\ln(x)]' = 1/x.$$

We can define the exponential function as the inverse function for the natural logarithm function:

$$\exp := [\ln]^{-1} : (-\infty, \infty) \to (0, \infty).$$

Now it is not difficult to prove all the standard properties of the exponential function.

Theorem Let $y, z \in \mathbb{R}$, then

1. $$\exp : (-\infty, \infty) \to (0, \infty)$$ is a bijection. It is strictly increasing.
2. $$\exp(y + z) = \exp(y) \exp(z).$$
3. If $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ then $$\exp\left(\frac{m}{n} y\right) = [\exp(y)]^{m/n}.$$
4. $$\exp(0) = 1.$$
5. $$[\exp(x)]' = \exp(x).$$

Definition We define e to be the number $\exp(1)$.

This definition implies $\ln(e) = 1$.

Let a be a positive real number and let $p \in \mathbb{R}$. Temporarily let us use the notation

$$E(a, p) := \exp(p \ln(a)).$$

If $p = m/n$ where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ then

$$E(a, p) = \exp(\frac{m}{n} \ln(a)) = [\exp(\ln(a))]^{m/n} = a^{m/n}.$$

This suggests that we define a^p as $E(a,p)$:

Definition. Let $a > 0$, and $p \in \mathbb{R}$ then we define

$$a^p := \exp(\ln(a)).$$

Exercise: Let $a > 0$. Find the derivative of $f(x) := a^x$ and $\int_p^q a^x \, dx$.

Exercise: Let $a > 1$ and let $\lambda(x) := a^x/[K + a^x]$ (a *logistic function*) . Show that λ is a bijection between $\mathbb{R}$ and $(0,1)$. Find its inverse function and find an antiderivative Λ, i.e. a function such that $\Lambda' = \lambda$.