Factoring $a^\mu - b^\mu$

Let a and b be positive numbers and let p and q be positive integers. A straight forward, but tedious, multiplication (or by using a computer algebra system such as *Mathematica* or *Maple* will show that

$$(a-b)\sum_{i=0}^{p-1} a^{i/q} b^{(p-1-i)/q} = (a^{p/q} - b^{p/q}) \sum_{j=0}^{q-1} a^{j/q} b^{(q-1-j)/q}.$$

Therefore, we have the factorization

$$a^{p/q} - b^{p/q} = (a-b)\frac{\sum_{i=0}^{p-1} a^{i/q} b^{(p-1-i)/q}}{\sum_{j=0}^{q-1} a^{j/q} b^{(q-1-j)/q}}. \tag{1}$$

Since a and b are positive numbers we can define

$$M := \max\{a, b, 1/a, 1/b\}.$$

Therefore

$$\left|\sum_{i=0}^{p-1} a^{i/q} b^{(p-1-i)/q}\right| \le pM^{(p-1)/q} \text{ and } \left|\sum_{j=0}^{q-1} a^{j/q} b^{(q-1-j)/q}\right| \ge q(1/M)^{(q-1)/q}.$$

Combining these inequalities with equation (1) we have

$$\left|a^{p/q} - b^{p/q}\right| \le \frac{p}{q} M^{[p+q-2]/q} |a-b|. \tag{2}$$

We have therefore established the following fact

Proposition. Let p and q be positive integers and let $M > 1$. Then there exists a positive number C such that for all $1/M \le a, b \le M$ we have

$$\left|a^{p/q} - b^{p/q}\right| \le C|a-b|$$

Note that once we have the *Mean Value Theorem* of Calculus this result becomes very easy to establish (an exercise for later!).