

## Review questions for chapter 0 with hints and some answers

1. Try to prove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

2. Prove that

$$A \setminus (B \setminus C) = (A \cap C) \cup (A \setminus B).$$

Hint:  $A \setminus (B \setminus C) = A \cap (B \cap C^c)^c$  and apply DeMorgan's Laws.

3. Is the composition of two injections an injection? Why?

Answer is yes: If  $g : A \rightarrow B$ ,  $f : B \rightarrow C$  and  $f(g(a)) = f(g(b)) \Rightarrow g(a) = g(b) \Rightarrow a = b$

4. Is the composition of two surjections a surjection? Why?

Answer is yes: If  $c \in C \Rightarrow \exists b \in B$  s.t.  $f(b) = c \Rightarrow \exists a \in A$  s.t.  $g(a) = b$ . Therefore  $f(g(a)) = c$ .

5. Let  $|C|$  denote the cardinality of the set  $C$ . Let  $A$  and  $B$  be sets. Explain carefully what is meant by each of these:  $|A| = |B|$ ,  $|A| \leq |B|$ ,  $|A| < |B|$ .

6. Show that the interval  $(0, 1)$  and  $\mathbb{R}$  are cardinally equivalent.

One needs to construct a bijection from  $\mathbb{R}$  to  $(0, 1)$ . For example  $y = 1/2 + (1/\pi) \arctan(x)$  or  $y = e^x/(1 + e^x)$ .

7. Show that the interval  $[0, 1]$  and  $\mathbb{R}$  are cardinally equivalent.

Use the previous problem and the injections  $[0, 1] \subset \mathbb{R}$  and  $(0, 1) \subset [0, 1]$  and the Cantor-Bernstein Theorem.

8. Let  $p \in \mathbb{N}$  and  $z > 0$ . Let

$$S := \{x \in \mathbb{R} \mid x^p \leq z, \quad x \geq 0\}.$$

(a) Show that  $S \neq \emptyset$ .

(b) Show that  $S$  is bounded from above by  $z + 1$ .

(c) Let  $q$  be the least upper bound of  $S$ . It can be shown that if  $q^p > z$  then for a sufficiently small number  $\delta > 0$  we have that  $q - \delta$  is an upper bound for  $S$ . It can also be shown that if  $q^p < z$  then for a sufficiently small number  $\delta > 0$  we have that  $(q + \delta)^p \leq z$ . Use these facts to prove that if  $q$  is the least upper bound of  $S$  then  $q^p = z$ , i.e.  $q = \sqrt[p]{z}$ .

This is a long problem. More difficult than anything you will be asked on an exam. The procedure follows that of the textbook (page 25) where the case  $p = 2$  is done. You can use the fact that for positive  $z$ , by the binomial theorem,  $(1 + z)^p > 1 + pz$ .

9. Let  $A$  and  $B$  be two finite sets,  $|A| = m$  and  $|B| = n$ , and let  $\mathfrak{F}(A, B)$  be the set of all functions from  $A$  to  $B$ . What is  $|\mathfrak{F}(A, B)|$  equal to? For each  $a \in A$  there are  $n$  possible values for  $f(a)$  and so the answer is  $n^m$ . What is the cardinality of the set  $\mathfrak{F}(\mathbb{N}, \{0, 1\})$ ? Answer:  $c$ . Think of a sequence of zeros and ones with a decimal point ahead of it, thus representing all numbers between 0 and 1 in binary form.

10. Suppose  $a < x < b$ . Show there is a number  $\epsilon > 0$  such that  $a < x - \epsilon < x < x + \epsilon < b$ . Answer: sketch to see why an answer is  $\epsilon = \min(b - x, x - a)$ .