Extra HW set 2 answers

1. Undetermined coefficients with $Y = C\, e^{i\omega t}$ yields

$$C = \frac{1}{170 - \omega^2 + 2\, i\, \omega}$$

and so

$$Y = \frac{\left[(170 - \omega)^2 - 2\, i\, \omega \right] e^{i\omega t}}{\left(170 - \omega^2\right)^2 + 4\, \omega^2}$$

or

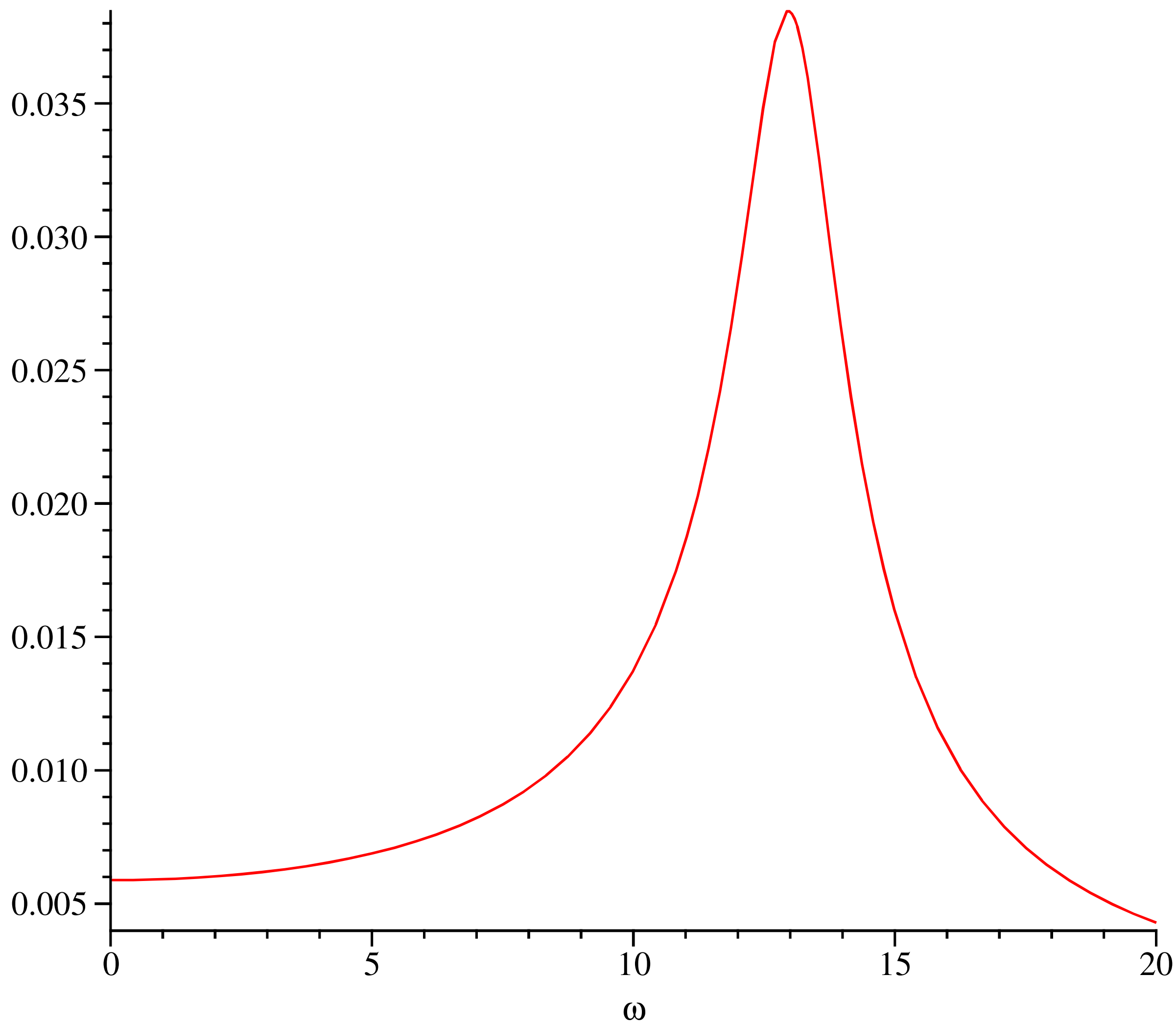
$$Y = \frac{\left[\left(170 - \omega^2\right) - 2\, i\, \omega \right] \left[\cos(\omega\, t) + i \sin(\omega\, t) \right]}{\left(170 - \omega^2\right)^2 + 4\, \omega^2}$$

2. Multiplying out the above expression and taking the imaginary part

$$v = \Im(Y) = \frac{\left(170 - \omega^2\right) \sin(\omega\, t) - 2 \cos(\omega\, t)}{\left(170 - \omega^2\right)^2 + 4\, \omega^2}$$

3. Let $A = \dfrac{1}{\sqrt{\left(170 - \omega^2\right)^2 + 4\, \omega^2}}$ and $\varphi = \arctan\left(\dfrac{2\, \omega}{170 - \omega^2} \right)$ then $v = A \sin\, (\omega\, t - \varphi)$.

4. `plot(1/sqrt((170-omega^2)^2+4*omega^2),omega=0..20);`



5. **B:=sqrt((170-omega^2)^2+4*omega^2);BP:=diff(B,omega);solve(BP=0, omega);**

$$\sqrt{28900 - 336\,\omega^2 + \omega^4}$$

$$\frac{1}{2}\,\frac{-672\,\omega + 4\,\omega^3}{\sqrt{28900 - 336\,\omega^2 + \omega^4}}$$

$$0,\ 2\sqrt{42},\ -2\sqrt{42} \tag{1}$$

Hence $\omega_{res} = 2\sqrt{42}$. This is about

evalf(2*sqrt(42));

$$12.96148140 \tag{2}$$

This agrees with the graph, which shows resonance at about 13. We can calculate $\omega_0 = \sqrt{170}$

$$\omega_0 = \sqrt{170} \tag{3}$$

evalf(sqrt(170));

$$13.03840481 \tag{4}$$

So all values are about 13.

6. The general solution to the homogeneous equation is obtained from the roots of the characteristic equation $r^2 + 2\cdot r + 170 = 0$. The roots are $-1 \pm 13\,i$ and so

$$y_h = \left[c_1\cos(13\cdot t) + c_2\sin(5\cdot t)\right]\cdot\exp(-t).$$

and therefore

$$y = \left[c_1\cos(13\cdot t) + c_2\sin(5\cdot t)\right]\cdot\exp(-t) + \frac{\left(170-\omega^2\right)\sin(\omega\,t) - 2\cos(\omega\,t)}{\left(170-\omega^2\right)^2 + 4\,\omega^2} =$$

$$y = \left[c_1\cos(13\cdot t) + c_2\sin(5\cdot t)\right]\cdot\exp(-t) + \frac{29\cdot\sin(5\,t) - 2\cos(5\,t)}{4225}$$

We can now solve for the unknown constants using the initial conditions:

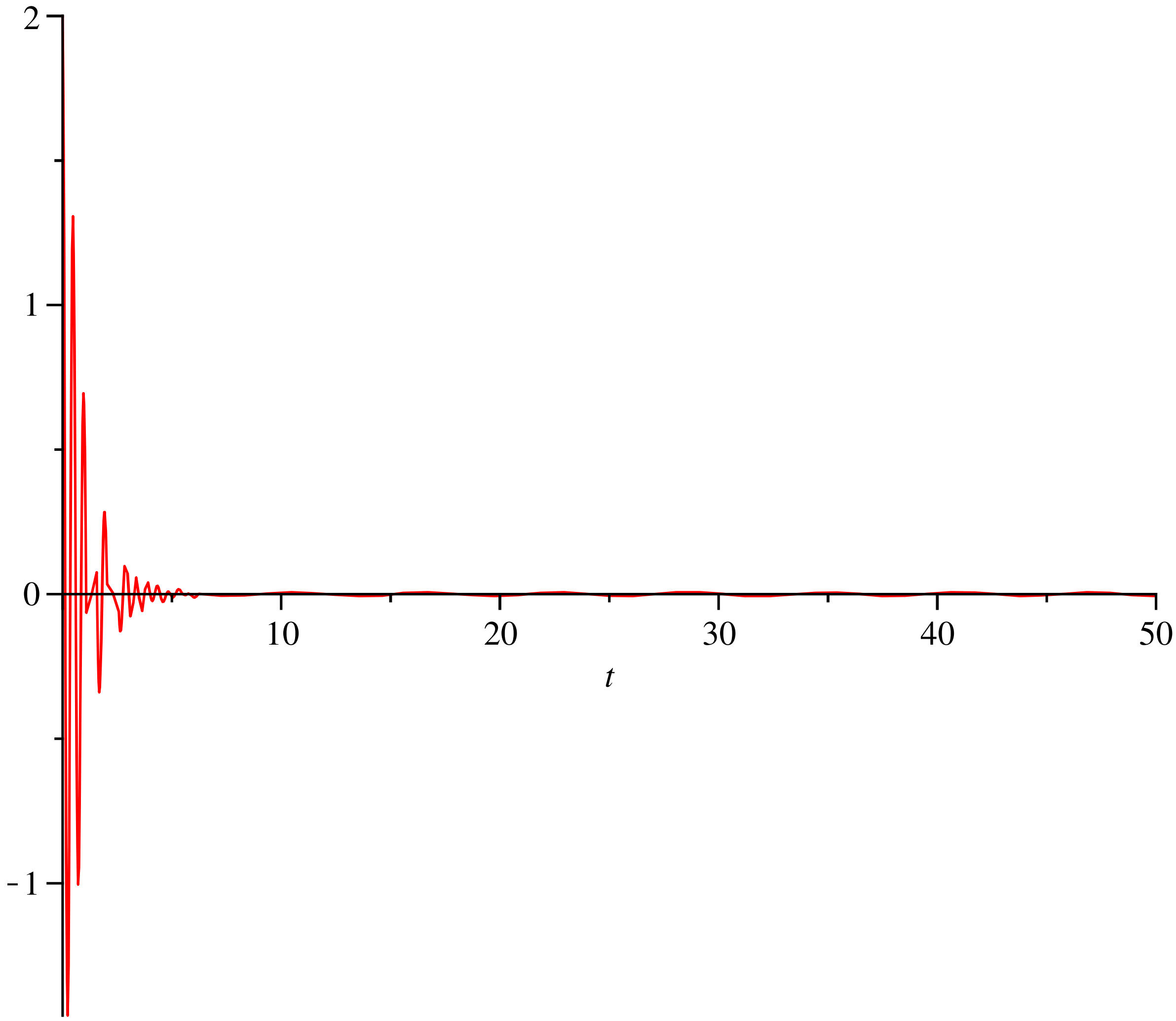
$$y = \left[\frac{8452}{4225}\cos(13\cdot t) + \frac{639}{4225}\sin(5\cdot t)\right]\cdot\exp(-t) + \frac{29\cdot\sin(5\,t) - 2\cos(5\,t)}{4225}$$

Plotting:

```
s:=((8452/4225)*cos(13*t)+(639/4225)*sin(5*t))*exp(-t)+(29*sin(5*t)-2*
cos(5*t))*(1/4225);
```

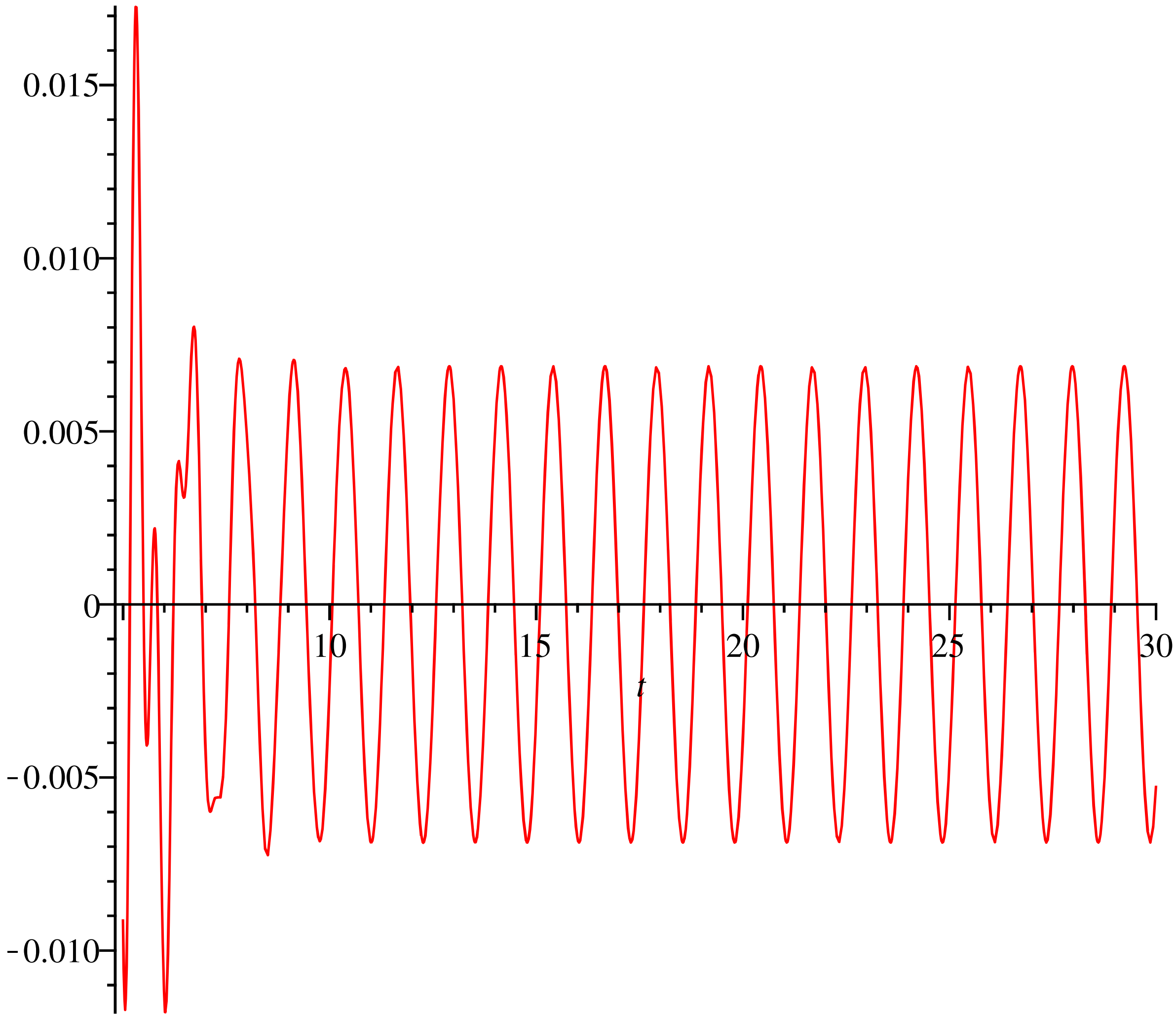
$$\left(\frac{8452}{4225}\cos(13\,t) + \frac{639}{4225}\sin(5\,t)\right)e^{-t} + \frac{29}{4225}\sin(5\,t) - \frac{2}{4225}\cos(5\,t) \qquad \textbf{(5)}$$

```
plot(s,t=0..50);
```



In order to better see the solution beyond t=5:

```
plot(s,t=5..30);
```


0.015
0.010
0.005
0
-0.005
-0.010
10
15
20
25
30
t