First order equations - generalities

Existence-Uniqueness Theorem *Let* $f(t,y)$ *and* $f_y(t,y)$ *be continuous functions in a an open set (domain)* $\mathscr{D} \subset \mathbb{R}^2$. *Let* (t_0, y_0) *be a point in* $\mathscr{D}$. *Then there exists an interval* (a,b) *with* $a < t_0 < b$ *such that the problem* $y' = f(t,y)$ *with initial condition* $y(t_0) = y_0$ *has a unique solution on the interval* (a,b).

Slope field The equation $y' = f(t,y)$ defines a slope field in the plane (or at least in the domain D of the function f). You should be able to sketch simple slope fields: where is the slope zero, where is it positive, and where is it negative? You should be able to sketch some typical solutions (trajectories).

Try for some simple functions, such as $f(t,y) = ty$, $f(t,y) = y\sin(t)$, $f(t,y) = t^2\sec(y)$.

First order autonomous equations

These are equations of the form $y' = f(y)$. If z is a root of f, i.e. $f(z) = 0$ then the constant function $y(t) \equiv z$ is a solution to the differential equation - called a *steady state solution.*

You should be able to sketch *phase line* diagrams for autonomous systems, i.e. the y-line (drawn horizontally) with its steady states marked and with arrows to indicate the direction of growth: $y' > 0 \Rightarrow \quad \rightarrow$ and $y' < 0 \Rightarrow \quad \leftarrow$

Try drawing some phase line diagrams. For example $f(y) = y^3 - 9y$, $f(y) = \cos(y)$, $f(y) = (y-1)(y-2)(y-3)(y-4)$

Applications - A

1. Population growth - exponential growth: $y' = ry$, $(\leftarrow 0 \rightarrow)$.
2. Population growth - logistic growth: $y' = ry(1 - y/K)$, $(\leftarrow 0 \rightarrow K \leftarrow)$.
3. Population growth - growth with critical threshold : $y' = ry(y - T)$, $(\rightarrow 0 \leftarrow T \rightarrow)$.
4. Population growth - logistic growth with critical threshold : $y' = ry(y-T)(1-y/K)$, $0 < T < K$, $(\rightarrow 0 \leftarrow T \rightarrow K \leftarrow)$.
5. Terminal velocity - $dv/dt = g - \gamma(v)$, where γ is an unbounded, monotone increasing function with $\gamma(0) = 0$.

Special first order equations

1. **Separable equations** $y' = g(t)f(y)$ The solution is obtained by integration :

$$\boxed{\int \frac{dy}{f(y)} = \int g(t)\,dt \; + \; \textit{constant}.}$$

2. **Linear first order equation** $y' + p(t)y = f(t)$. The solution is obtained by first finding an integrating factor:

$$\boxed{\mu(t) := \exp\left(\int p(t)\,dt\right)}$$

then multiplication by the integrating factor yields

$$y'(t)\mu(t) + y(t)p(t)\mu(t) = (y(t)\mu(t))' = f(t)\mu(t),$$

so that

$$\boxed{\mu(t)y(t) = \int^t f(\tau)\mu(\tau)\,d\tau \;+\; \textit{constant}.}$$

If an initial condition is given at t_0 then we may choose to do a definite integral:

$$\boxed{\mu(t)y(t) - \mu(t_0)y(t_0) = \int_{t_0}^t f(\tau)\mu(\tau)\,d\tau,}$$

and solve for $y(t)$.

Remarks

♣ Equations of the form $y' + p(t)y = 0$ maybe solved both ways: as separable equations, and as linear equations.

♠ Sometimes we can guess a particular solution to a linear equation : Suppose we have found a particular solution $v' + p(t)v = f(t)$. Now solve the much easier equation $u' + p(t)u = 0$. If we have found a nontrivial solution u then Cu is a solution for any constant C. Moreover, the general solution to $y' + p(t)y = f(t)$ is : $y(t) = v(t) + Cu(t)$. The value for C may be found from the initial condition.

Second order equations - generalities

Existence-Uniqueness Theorem *Let $f(t, y, z)$, $f_y(t, y, z)$, $f_z(t, y, z)$ be continuous functions in a an open set (domain) $\mathscr{D} \subset \mathbb{R}^3$. Let (t_0, y_0, y_1) be a point in $\mathscr{D}$. Then there exists an interval (a, b) with $a < t_0 < b$ such that the problem $y' = f(t, y, y')$ with initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_1$ has a unique solution on the interval (a, b).*

Special nonlinear second order equations

1. $$y'' = f(t, y')$$

   This is simply a first order equation for y', so let $u := y'$ and solve the first order equation $u' = f(t, u)$. Then $y(t) = \int^t u(\tau)\,d\tau + \textit{constant}$.

2. $$y'' = f(y)$$

   Find an antiderivative of f. Say that $F' = f$. Then write the equation as

   $$\frac{1}{2}\left([y']^2\right)' = (F(y))'.$$

   The solution is

   $$y' = \pm\sqrt{C_1 + 2F(y)},$$

   where C_1 is an arbitrary constant. This first order equation may be solved by separation of variables. Implicitly, the solution is given by

   $$t = \pm\int^y \frac{d\eta}{\sqrt{C_1 + 2F(\eta)}} \ + \ C_2,$$

   where C_2 is another arbitrary constant.

Linear second order equations

Reduction of order Suppose we know one solution $y_1(t)$ to the equation $y'' + p(t)y' + q(t)y = f(t)$, then we can find the general solution as follows: Write write $y(t) = u(t)y_1(t)$ and plug this into the equation, we get

$$y_1u'' + [2y_1' + py_1]u' = f.$$

Letting $v := u'$ we have a first order equation that can be solved for v:

$$y_1v' + [2y_1' + py_1]v = f.$$

Then $u = \int v(t)\,dt$.

Now we start with the most special and conclude with the theory. Let a, b, and c denote constants.

Homogeneous equations

1. $ay'' + by' + cy = 0$ - characteristic equation: $ar^2 + br + c = 0$
   (a) The characteristic equation has two distinct real roots r_1 and r_2. The general solution is

   $$\boxed{y = C_1\mathrm{e}^{r_1t} + C_2\mathrm{e}^{r_2t}.}$$

   (b) The characteristic equation has a repeated root r. The general solution is

   $$\boxed{y = [C_1 + C_2t]\mathrm{e}^{rt}.}$$

   (c) The characteristic equation has two distinct complex roots $\alpha \pm i\beta$. The general solution is

   $$\boxed{y = [C_1\cos(\beta t) + C_2\sin(\beta t)]\mathrm{e}^{\alpha t}.}$$

2. $at^2y'' + bty' + cy = 0$ (*Euler equation* - characteristic equation $ar(r-1) + br + c = 0$.
   (a) The characteristic equation has two distinct real roots r_1 and r_2. The general solution is

$$\boxed{y = C_1 t^{r_1} + C_2 t^{r_2}.}$$

(b) The characteristic equation has a repeated root r. The general solution is

$$\boxed{y = [C_1 + C_2 \ln(t)]t^r.}$$

(c) The characteristic equation has two distinct complex roots $\alpha \pm i\beta$. The general solution is

$$\boxed{y = [C_1 \cos(\beta \ln(t)) + C_2 \sin(\beta \ln(t))]e^{\alpha t}.}$$

Nonhomogeneous Equations

The general solution of a linear nonhomogeneous solution = a particular solution + the general solution to the associated homogeneous equation.

We have the following 3 methods for finding particular solutions:

1. **Undetermined coefficients** An expression of type (α, β) is a term that looks like $p(t)e^{\alpha t}\cos(\beta + q(t)e^{\alpha t}\sin(\beta t)$ where p and q are polynomial. The degree of the expression is the maximum of the degrees of p and q. Note that if $\beta = 0$ there remains only one term, $p(t)e^{\alpha t}$. If both α and β are zero then we are simply left with the polynomial p. *Example* $(3t^2 - 4t + 7)e^{-4t}\cos(7t)$ is a term of type $(-4, 7)$ and it is of degree 2.
   Example $(t^3)e^{-4t}$ is a term of type $(-4, 0)$ and it is of degree 3.
   Example $\sin(7t)$ is a term of type $(0, 7)$ and it is of degree 0.
   Example $(t^4 + 4t^2 + 6)$ is a term of type $(0, 0)$ and it is of degree 2.

   $$ay'' + by' + cy = f(t), \quad f \text{ is of type } (\alpha, beta) \text{ with degree } n.$$

   In this case follow the following recipe:

   (a) Find the roots of the characteristic equation and get the general solution $y_h(t)$ to the associated homogeneous equation. This general solution will contain two arbitrary constants.

   (b) Let y be a general term of type (α, β) and degree n all multiplied by t^s where s is chosen as follows:

      i. If $\alpha + i\beta$ is not a root of the characteristic equation, then $s = 0$.

      ii. If $\alpha + i\beta$ is a simple root of the characteristic equation, then $s = 1$.

      iii. If $\alpha + i\beta$ is not a double root of the characteristic equation, then $s = 2$.

   (c) Plug this expression for y into the differential equation and find the coefficients of y that will give a particular solution y_p.

   (d) The general solution to the nonhomogeneous equation is $y(t) = y_h(t) + y_p(t)$.

   (e) If there are any initial conditions to satisfy, impose them on y to evaluate the two remaining undetermined coefficients.

2. **Variation of Parameters**. Make sure the differential equation is in standard form : $y'' + py' + qy = g(t)$. Find two linearly independent solutions y_1 and y_2 to the associated homogeneous equation. The particular solution will have the form $y = uy_1 + vy_2$ where u and v must satisfy:

   $$y_1 u' + y_2 v' = 0$$
   $$y_1' u' + y_2' v' = g.$$

Solve this linear system for u' and v' and then

$$y(t) = y_1(t)\int u'(t)\,dt + y_2(t)\int v'(t)\,dt.$$

3. **Duhamel's Principle** This is sometimes useful to know. Make sure the differential equation is in standard form : $y'' + py' + qy = g(t)$. Find the solution v to the initial value problems

   $$v'' + pv' + qv = 0, \quad v(0) = 0, \quad v'(0) = 1.$$

   Then the following expression provides a particular solution:

   $$y_p = \int_0^t v(t-s)g(s)\,ds.$$

   If we also solve

   $$u'' + pu' + qu = 0, \quad u(0) = 1, \quad u'(0) = 0.$$

   then

   $$y_0 u(t) + y_1 v(t) + \int_{t_0}^t v(t-s)g(s)\,ds$$

   is the unique solution of the initial value problem

   $$y'' + py' + qy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1.$$

Applications - B

Oscillations - mechanical, electrical, biological. Terminology Let A, ω and α be constants and consider $A\cos(\omega t - \alpha)$. A is called the amplitude, ω is the angular frequency, $T := 2\pi/\omega$ is the period, and $\nu := \omega/(2\pi)$ is the frequency. Note $\nu = 1/T = \omega/(2\pi)$.

$$ay'' + by' + cy = F\cos(\Omega t - \phi),$$

where a, b, c, A, Ω and ϕ are constants, $a > 0$, $c > 0$, and $b \geq 0$.

1. **Free oscillations without dissipation**: $F = 0$, $b = 0$.

   $$y = A\cos(\omega_0 t - \theta), \quad \omega_0 = \sqrt{\frac{c}{a}},$$

   where A and θ are arbitrary constants.

2. **Free oscillations with dissipation**, $F = 0$,$b > 0$:

   $$y = A\mathrm{e}^{-bt/2a}\cos(\omega t - \theta), \quad \omega = \sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}, \quad \omega^2 = \omega_0^2\left(1 - \frac{b^2}{4ac}\right).$$

3. **Forced oscillations** $F \neq 0$. We embed the problem into a differential equation for a complex-valued solution:

   $$aY'' + bY' + cY = A\mathrm{e}^{i(\Omega t - \phi)}, \quad Y = y + iz.$$

   Now $Y = A\mathrm{e}^{-i\phi}U$ where

   $$aU'' + bU' + cU = \mathrm{e}^{i\Omega t}.$$

(a) $b \neq 0$. Undetermined coefficients yields the particular solution

$$U = \frac{\mathrm{e}^{i\Omega t}}{c - a\Omega^2 + bi\Omega}$$

Hence

$$Y = \frac{A\mathrm{e}^{i(\Omega t - \phi)}}{c - a\Omega^2 + bi\Omega} = \frac{A[c - a\Omega^2 - bi\Omega]\mathrm{e}^{i(\Omega t - \phi)}}{(c - a\Omega^2)^2 + b^2\Omega^2}$$

Let δ be the angle such that

$$\sin(\delta) = \frac{b\Omega}{\sqrt{(c - a\Omega^2)^2 + b^2\Omega^2}}$$

then

$$c - a\Omega^2 + bi\Omega = \sqrt{(c - a\Omega^2)^2 + b^2\Omega^2}\mathrm{e}^{i\delta},$$

then

$$Y = \frac{A\mathrm{e}^{i(\Omega t - \phi - \delta)}}{\sqrt{(c - a\Omega^2)^2 + b^2\Omega^2}}.$$

Since y is the real part of Y

$$y(t) = \frac{A\cos(\Omega t - \phi - \delta)}{\sqrt{(c - a\Omega^2)^2 + b^2\Omega^2}}.$$

Remarks:

i. The solution lags behind the forcing term by a *phase shift* δ.

ii. The *response* ρ is the ratio of the solution amplitude to the amplitude of the forcing term:

$$\rho = \frac{1}{\sqrt{(c - a\Omega^2)^2 + b^2\Omega^2}}.$$

Defining $\Gamma = b^2/(ac)$ we can write

$$\rho = \frac{1}{c\sqrt{[1 - (\Omega/\omega_0)^2]^2 + \Gamma(\Omega/\omega_0)^2}}.$$

and, considering this as a function of the angular frequency Ω, we see that it has a peak at some value Ω_{max}. This is the angular frequency at which the system is at *resonance.* Minimizing the denominator we can compute

$$\Omega_{max}^2 = \omega_0^2\left(1 - \frac{b^2}{2ac}\right).$$

(b) Special case where b is very small:

$$\Omega_{max} \approx \omega \approx \omega_0, \quad \rho_{max} \approx \frac{1}{b\omega_0}.$$

(c) Special case $b = 0$, with $\Omega \neq \omega_0$:

$$y(t) = \frac{F\cos(\Omega t - \phi)}{a[\omega_0^2 - \Omega^2]}.$$

(d) Special case $b = 0$, with $\Omega = \omega_0$: In this case a particular solution is provided by

$$y(t) = \frac{Ft\sin(\omega_0 t - \phi)}{\sqrt{4ac}}.$$

This solution grows without bound.

Systems of differential equations

1. **Reduction to a first-order system**

   (a) $y'' = f(t, y, y')$. Let $u_1 = y$, $u_2 = y'$, then

   $$\begin{aligned} u_1' &= u_2 \\ u_2' &= f(t, u_1, u_2) \end{aligned}$$

   (b) $y'' = f(t, y, y', z)$ and $z' = g(t, y, y', z)$. Let $u_1 = y$, $u_2 = y'$, and $u_3 = z$, then

   $$\begin{aligned} u_1' &= u_2 \\ u_2' &= f(t, u_1, u_2, u_3) \\ u_3' &= g(t, u_1, u_2, u_3) \end{aligned}$$

   (c) $y'' = f(t, y, y', z, z')$ and $z'' = g(t, y, y', z, z')$. Let $u_1 = y$, $u_2 = y'$, $u_3 = z$, and $u_4 = z'$ then

   $$\begin{aligned} u_1' &= u_2 \\ u_2' &= f(t, u_1, u_2, u_3, u_4) \\ u_3' &= u_4 \\ u_4' &= g(t, u_1, u_2, u_3, u_4) \end{aligned}$$

2. **First order homogeneous linear systems with 2 unknowns**

   $$\mathbf{y}' = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{A}\mathbf{y}.$$

   Characteristic equation:

   $$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+c)\lambda + (ac - bd) = 0$$

   (a) Two real roots λ_1 and λ_2 with corresponding eigenvectors $\mathbf{u}$ and $\mathbf{v}$:
   General solution $\mathbf{y} = c_1 e^{\lambda_1 t}\mathbf{u} + c_2 e^{\lambda_2 t}\mathbf{v}$

   (b) Two complex roots $\lambda = \alpha \pm i\beta$ with corresponding eigenvectors $\mathbf{a} \pm i\mathbf{b}$.
   Let

   $$\begin{aligned} \mathbf{u} &= \Re\left(\mathbf{a}e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)\right) \\ \mathbf{v} &= \Im\left(\mathbf{a}e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)\right) \end{aligned}$$

   General solution is $\mathbf{y} = c_1\mathbf{u} + c_2\mathbf{v}$

   (c) Repeated eigenvalue (But $\mathbf{A}$ not a multiple of the identity). Keeping in mind what happens in case of a repeated root for the characteristic equation of a second order linear equation with constant coefficients we suspect there may be a solution of the form $(\mathbf{u} + t\mathbf{v})e^{\lambda t}$. Plug this into the equation $\mathbf{y}' = \mathbf{A}\mathbf{y}$ we get $[\mathbf{v} + \lambda(\mathbf{u} + t\mathbf{v})]e^{\lambda t} = (\mathbf{A}\mathbf{u} + t\mathbf{A}\mathbf{v})e^{\lambda t}$. This is satisfied if $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ and $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{v}$. Hence

      i. Find an eigenvector $\mathbf{v}$.
      ii. Solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{v}$ for $\mathbf{u}$.
      iii. The general solution is $\mathbf{y} = [c_1\mathbf{v} + c_2(\mathbf{u} + t\mathbf{v})]e^{\lambda t}$.

Remark These ideas extend to n equations in n unknowns, but the linear algebra gets awfully complicated and you need to study diagonalization of square matrices and the *Jordan canonical form.* Thankfully, this is not within the scope of a first course in differential equations!

Numerical Solutions

1. **Euler method for scalar first order equations**

$$y' = f(t,y), \quad y(t_0) = y_0 \text{ Euler method: } t_{n+1} = t_n + h, \quad y_{n+1} y_n + hf(t_n, y_n).$$

Often the notation $f_n := f(t_n, y_n)$ is used.

Example $y' = t^2 + y^2$, $y(0) = 1$.

n	t_n	y_n	f_n	hf_n	$y_n + hf_n$
0	0	1	1	0.1	1.1
1	0.1	1.1	1.220	0.122	1.222
2	0.2	1.222	1.493	0.149	1.371
3	0.3	1.371			

Note that this computation tells us that $y(0.3) \approx 1.371$.

2. **Euler method for first order systems**

$$y' = f(t,y,z), \quad z' = g(t,y,z) \quad y(t_0) = y_0, \quad z(t_0) = z_0$$

$$\text{Euler method: } t_{n+1} = t_n + h,\ y_{n+1} = y_n + hf(t_n, y_n, z_n),\ z_{n+1} = z_n + hg(t_n, y_n, z_n).$$

Notation $f_n := f(t_n, y_n)$ and $g_n := g(t_n, y_n)$.

Example $y' = t^2 + y^2 + z^2$, $y(0) = 1$, $z' = 1 + t + yz$

n	t_n	y_n	z_n	f_n	g_n	hf_n	hg_n	$y_n + hf_n$	$z_n + hg_n$
0	0	1	-1	2	0	0.2	0	1.2	-1
1	0.1	1.2	-1	2.45	-0.1	0.245	-0.1	1.445	-1.01
2	0.2	1.445	-1.01	3.148	-0.259	0.315	-0.0259	1.760	-1.036
3	0.3	1.760	-1.036						

Note that this computation tells us that $y(0.3) \approx 1.760$ and $z(0.3) \approx -1.036$.

Laplace Transform

1. **Definition.** Let $\mathscr{E}$ denote the set of all piecewise continuous (possibly complex-valued) functions on $[0, \infty)$ that are of at most exponential growth. That is to say, a piecewise continuous function $f : [0, \infty) \to \mathbb{C}$ is a member of $\mathscr{E}$ only if there exist positive numbers M and a such that $|f(t)| \leq M\mathrm{e}^{at}$. Let $\mathscr{F}$ denote the set of all functions F that have a domain of the form $(a, \infty$ for some number a and such that $F : (a, \infty) \to \mathbb{C}$ is continuous. The Laplace transform $\mathcal{L} : \mathscr{E} \to \mathscr{F}$ is defined by:

$$\mathcal{L}[f] = F \text{ if } F(s) = \int_0^\infty f(t)\, \mathrm{e}^{-st}\, dt.$$

Example The following function is a member of $\mathscr{E}$:

$$f(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < 2 \\ 1 & \text{if} \quad 2 \leq t < 3 \\ \mathrm{e}^{4t} & \text{if} \quad 3 \leq t \end{cases} \tag{1}$$

Example Let k be some real number. The following function is a member of $\mathscr{E}$:

$$F(s) = \frac{1}{s-k},$$

since it is continuous on the domain (k, ∞).

Example Let f be given by equation (1), then we easy compute its Laplace transform:

$$F(s) = \int_0^2 0\,\mathrm{e}^{-st}\,dt + \int_2^3 1\,\mathrm{e}^{-st}\,dt + \int_0^2 \mathrm{e}^{4t}\,\mathrm{e}^{-st}\,dt =$$

$$\frac{\mathrm{e}^{-2s} - \mathrm{e}^{-3s}}{s} - \frac{\mathrm{e}^{3(4-s)}}{4-s}.$$

Note that $f(t) \le \mathrm{e}^{4t}$ and F has domain $(4, \infty)$.

2. **Partial fractions for finding the inverse Laplace transform**. Consider

$$F(s) = \frac{ps+q}{s^2+bs+c}.$$

   (a) In case the denominator can be factored over the reals: $s^2 + bs + c = (s-\alpha)(s-\beta)$. then

$$F(s) = \frac{c_1}{s-\alpha} + \frac{c_2}{s-\beta}$$

   and hence $f(t) = c_1\mathrm{e}^{\alpha t} + c_2\mathrm{e}^{\beta t}$.

   (b) In case the denominator cannot be factored over the reals, we complete the square: $s^2 + bs + c = (s + b/2)^2 + d^2$, where $d^2 = c - b^2/4$ (note that if $c - b^2/4 < 0$ then the quadratic can be factored over the reals). Therefore

$$F(s) = \frac{ps+q}{(s+b/2)^2+d^2} = \frac{p(s+b/2) + (q - pb/2)}{(s+b/2)^2+d^2}.$$

   Therefore

$$f(t) = p\mathrm{e}^{-bt/2}\cos(dt) + \frac{q-b/2}{d}\mathrm{e}^{-bt/2}\sin(dt).$$

3. **Solving differential equations with Laplace transforms** If we denote the Laplace transform of $y(t)$ by $Y(s)$ and the Laplace transform of $f(t)$ by $F(s)$ then $\mathcal{L}[y'] = sY(s) - y(0)$ and $\mathcal{L}[y''] = s^2Y(s) - sy(0) - y'(0)$ and therefore linear ODEs with constant coefficients can be Laplace transformed:

$$ay' + by = f(t), \quad y(0) = y_0 \Rightarrow a[sY - y_0] + bY = F(s),$$

   and $Y(s)$ computed (easy). Then we try to find the inverse Laplace transform of $Y(s)$, which in general is not so easy.

$$ay'' + by' + cy = f(t), \quad y(0) = y_0,\ y'(0) = y_1 \Rightarrow a[s^2Y - y_0s - y_1] + b[sY - y_0] + Y = F(s).$$

   $Y(s)$ is then computed and then we try to find the inverse Laplace transform of $Y(s)$.

   ♣ An equation is **either** in the time domain (in t) **or** in the “frequency” domain (in s). If you have both s and t in the same equation, you have probably made a mistake.

   ♠ To find the inverse Laplace transform of Y you usually do partial fractions or use the convolution theorem (see below).

4. **The singularities of F determine the stability of f .**

Example. The usefulness of the Laplace transform lies less in its usefulness for explicitly computing solutions than in its usefulness in determining the behavior of solutions without explicitly computing them. For example, suppose that

$$Y(s) = \frac{1}{(s+1)^2(s-3)}$$

then it has a simple pole at 3 and a double pole at -1. Therefore $y(t)$ will contain a term that behaves like e^{3t} and a term that behaves like $t\mathrm{e}^{-t}$.

Example. Suppose that

$$Y(s) = \frac{1}{(s^2 +^4 s + 13)(s+4)}$$

then it has a simple poles at -4 and at $-2 \pm 3i$. Therefore $y(t)$ will contain a term that behaves like e^{4t} and sinusoidal terms that die exponentially like e^{-2t}. The sinusoids in this case have angular frequency 3.

Example. Suppose that

$$Y(s) = \frac{5 - 5\mathrm{e}^{-s}}{s^2}.$$

Using the Taylor expansion for the exponential, and canceling a factor of s we get

$$Y(s) = \frac{5 - 5s/2 + 5s^2/6 - \cdots}{s},$$

which has a simple pole $5/s$ at the origin. This indicates that $y(t) \to 5$ as $t \to \infty$. In particular we can determine stability: if all the poles of $Y(s)$ have negative real part, then $y(t) \to 0$ as $t \to \infty$.

5. **The Convolution Theorem.** Let $f, g : [0, \infty)$, then their convolution is the function $h = f * g$ where

$$h(t) = \int_0^t f(t-s)g(s)\,ds.$$

The important properties of the convolution are

♣ $u * v = v * u$

♠ $(u * v) * w = u * (v * w)$

♢ $\mathcal{L}[u * v] = \mathcal{L}[u]\mathcal{L}[v]$

Example

$$\mathcal{L}^{-1}\left[\frac{1}{(4+s^2)(1+s^2)}\right] = \int_0^t \frac{1}{2}\sin(2(t-s))\sin(s)\,ds = -\frac{1}{6}\sin(2t) + \frac{1}{3}\sin(t).$$

Application Consider the equation $ay'' + by' + cy = f(t)$ and let y_p be the solution with initial conditions $y_p(0) = 0$ and $y_p'(0) = 0$. Then $Y_p(s) = [as^2 + bs + c]^{-1}[F(s)$. Let v be the solution of the associated homogeneous equation with initial conditions $v(0) = 0$, $v'(0) = 1/a$. Then $V(s) = [as^2 + bs + c]^{-1}$. That is to say, $Y_p(s) = V(s)F(s)$. Then by the convolution theorem $y_p = v * f$. Hence

$$y_p(t) = \int_0^t v(t-s)f(s)\,ds.$$

Example Let ω be a positive number and consider the problem

$$y'' + \omega^2 y = f(t),\ y(0) = y_0,\ y'(0) = y_1.$$

Let $u(t) = \cos(\omega t)$ and $v(t) = \omega^{-1}\sin(\omega t)$. Note that $u(0) = 1$, $u'(0) = 0$, $v(0) = 0$, and $v'(0) = 1$. Applying the above we see that the solution to the above nonhomogeneous initial value problem is given by

$$y(t) = y_0\cos(\omega t) + y_1\frac{\sin(\omega t)}{\omega} + \int_0^t \frac{\sin(\omega(t-s))}{\omega}f(s)\,ds.$$

Exercise Obtain a similar formula for the equation $y'' - \omega^2 y = f(t)$.

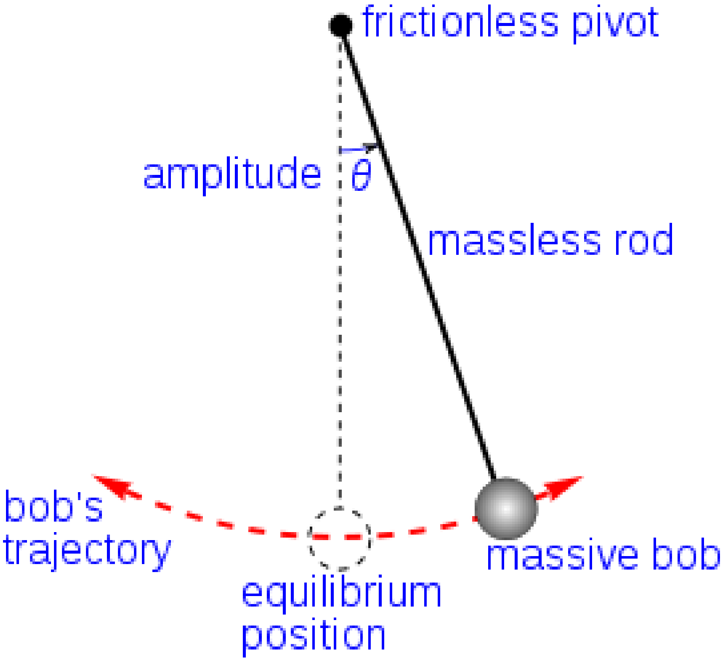

Figure 1: Simple pendulum

Applications - C

1. **The period of a pendulum as a function of amplitude - a nonlinear second order equation**. One of the most basic models in dynamics, is that of the swinging pendulum. Let g denote the acceleration due to gravity and let ℓ denote the length of the pendulum (distance between the fulcrum and the center of mass). If we denote by θ the angular displacement from equilibrium, then the equation for the swinging pendulum becomes
$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin(\theta). \tag{2}$$
For convenience define $\omega_0 := \sqrt{g/\ell}$, so that the differential equation becomes $\theta'' = -\omega_0^2\sin(\theta)$. Typically, one makes the "small angle" approximation, meaning that we use the fact that if θ is small then $\sin\theta \approx \theta$ and hence
$$\theta'' = -\omega_0^2\theta, \quad |\theta| \ll \pi. \tag{3}$$
This has the solution $\theta = A\cos(\omega_0 t - \phi)$, where the amplitude A and the phase angle ϕ are the arbitrary constants that may be found from the initial conditions. The number ω_0 is called the *angular frequency*. The *period* must satisfy $\omega_0 T = 2\pi$, and hence $T = 2\pi/\omega_0$. The *frequency* f is equal to $1/T$ and hence $f = \omega_0/(2\pi)$.

   Although we can't get a closed form solution to equation (2) in terms of elementary functions, we can find an expression for the period as a function of the amplitude. The amplitude is the positive angle θ_0 where the pendulum reverses direction, i.e. it the value of θ at a point where $\theta' = 0$. Multiplying equation (3) by θ' and writing g/ℓ as ω_0^2:
$$\theta''\theta' = -\omega_0^2\sin(\theta)\theta'.$$
But this is the same as
$$\frac{1}{2}\left([\theta']^2\right)' = \omega_0^2[\cos(\theta)]'.$$
We can integrate this
$$\frac{1}{2}[\theta']^2 = \omega_0^2\cos(\theta) + c_1.$$
But using the fact that $\theta' = 0$ when $\theta = \theta_0$ we have that $c_1 = -\omega_0^2\cos(\theta_0)$. Therefore
$$\theta' = \pm\sqrt{2}\omega_0\sqrt{\cos(\theta) - \cos(\theta_0)}. \tag{4}$$

During half the period, as the pendulum swings from $-\theta_0$ to $+\theta_0$, we have $\theta' > 0$ then during the next half period $\theta' < 0$. Separating variables we can solve equation (4) for the time interval that θ is increasing:

$$\int_0^\theta \frac{dy}{\sqrt{\cos(y) - \cos(\theta_0)}} = \sqrt{2}\omega_0 t + c_2.$$

Since $T/4$ is the time it takes for the pendulum to travel from 0 to θ_0 we have

$$T = T(\theta_0) = \frac{2\sqrt{2}}{\omega_0} \int_0^{\theta_0} \frac{dy}{\sqrt{\cos(y) - \cos(\theta_0)}}.$$

We can show that the function $T(\theta_0)$ approaches $2\pi/\omega_0$ as θ_0 decreases to zero. This was expected, of course. But also, we can show that as $\theta_0 \to \pi$ we have $T(\theta_0) \to \infty$. Here are the arguments:

(a) Small amplitudes. We can use the approximation $\cos\theta \approx 1 - \theta^2/2$ for small θ so that

$$T \approx \frac{4}{\omega_0} \int_0^{\theta_0} \frac{dy}{\sqrt{\theta_0^2 - y^2}}.$$

Making the substitution $y = \theta_0 \sin x$ this becomes

$$T \approx \frac{4}{\omega_0} \int_0^{\pi/2} \frac{\cos(x)dx}{\sqrt{1 - \sin^2(x)}} = \frac{4}{\omega_0} \int_0^{\pi/2} 1\, dx = \frac{2\pi}{\omega_0}.$$

(b) Large amplitudes. Using the fact that for $\theta_0 \approx \pi$ we have $1 \geq -\cos(\theta_0) \approx 1$ we see that

$$T = \frac{2\sqrt{2}}{\omega_0} \int_0^{\theta_0} \frac{dy}{\sqrt{\cos(y) - \cos(\theta_0)}} \geq \frac{2\sqrt{2}}{\omega_0} \int_0^{\theta_0} \frac{dy}{\sqrt{\cos(y) + 1}}.$$

Using the trigonometric identity $1 + \cos(y) = 2\cos^2(\theta/2)$ this last integral becomes

$$\frac{2}{\omega_0} \int_0^{\theta_0} \frac{dy}{\cos(y/2)} = \frac{2}{\omega_0} \int_0^{\theta_0} \sec(y/2)\, dy = \frac{4}{\omega_0} \ln\left[\tan(\theta_0/2) + \sec(\theta_0/2)\right].$$

As θ_0 increases towards π, both $\tan(\theta_0/2)$ and $\sec(\theta_0/2)$ become arbitrarily large, so

$$\lim_{\theta_0 \uparrow \pi} T(\theta_0) = \infty.$$

Figure 2: Catenary

2. **The catenary - a boundary value problem**. We derive the equation for the curve taken on by a hanging cable. It is assumed that only force exerted on the cable is its own weight. Suppose that the cable has length $2L$ and is suspended from the points with coordinates $(\pm\ell, 0)$. Let s denote the arc length along the cable, $-L \leq s \leq L$, and let $(x(s), y(s))$ be the position of the point (denoted $P(s)$) at a distance s along the cable from its midpoint. Let $T(s)$ denote the tension in the cable at the point $P(s)$ and let $\theta(s)$ denote the angle that the tangent to the cable at the point $P(s)$ makes with the horizontal (hence $dy/dx = \tan\theta$). The component T_0 of the tension in the horizontal direction must be constant (otherwise there would be horizontal movement): $T(s)\cos(\theta(s)) = T_0$. Moreover $x' = dx/ds = \cos(\theta(s))$, $y' = dy/ds = \sin(\theta(s))$, $(dx/ds)^2 + (dy/ds)^2 = 1$, and therefore

   $$\tan(\theta(s)) = \frac{\pm y'}{\sqrt{1-(y')^2}}.$$

   Consider the piece of cable between s and $s + \Delta s$. For definiteness we assume $s > 0$, so $\tan(\theta(s)) > 0$. The net vertical force on this piece due to tension supports the weight of that piece of cable, hence

   $$T(s+\Delta s)\sin(\theta(s+\Delta s)) - T(s)\sin(\theta(s)) = \rho\Delta s\, g,$$

   and this yields

   $$\frac{d}{ds}\left[T(s)\sin(\theta(s))\right] = \rho g.$$

   Let $\gamma := T_0/(\rho g)$, then the above equation may be written as

   $$\frac{d}{ds}\left[\frac{dy/ds}{\sqrt{1-(dy/ds)^2}}\right] = 1/\gamma.$$

   Let $v = y'$, then this is a simple first order equation for v that yields

   $$\frac{v}{\sqrt{1-v^2}} = s/\gamma - c_1.$$

   By symmetry, $v = 0$ when $s = 0$, i.e. at the midpoint the slope is zero, and so $c_1 = 0$. Solving for v yields

   $$y' = \frac{s}{\gamma^2 + s^2}.$$

   We can integrate this

   $$y(L) - y(s) = \int_s^L \frac{s}{\gamma^2+s^2}\, ds = \sqrt{\gamma^2 + L^2} - \sqrt{\gamma^2 + s^2}.$$

Since $y(L) = 0$ we have

$$y(s) = \sqrt{\gamma^2 + s^2} - \sqrt{\gamma^2 + L^2}.$$

Next we find $x(s)$:

$$x' = \sqrt{1 - (y')^2} = \frac{\gamma}{\sqrt{\gamma^2 + s^2}}.$$

Make the substitution $s = \gamma \sinh(\sigma)$ we have

$$x(s) - x(0) = \int_0^s \frac{\gamma}{\sqrt{\gamma^2 + s^2}}\, ds = \int_0^{\sinh^{-1}(s/\gamma)} \frac{\gamma^2 \cosh(\sigma)\, d\sigma}{\sqrt{\gamma^2 + \gamma^2 \sinh(\sigma)^2}} = \gamma \sinh^{-1}(s/\gamma).$$

Therefore $s = \gamma \sinh(x/\gamma)$ and substituting this into the above expression for y:

$$y = \gamma \left[\sqrt{1 + \sinh^2(x/\gamma)} - \sqrt{1 + \sinh^2(L/\gamma)} \right].$$

One more thing! We need to find the constant γ. Presumably we are given the values for ρ and g, but we don't know T_0. However, $x(L) = \ell$, and therefore can know γ from solving:

$$L = \gamma \sinh(\ell/\gamma). \tag{5}$$

Exercise. Show that if $L > \ell$ then equation (5) has a unique positive solution γ. Hint: consider the equation $\kappa z = \sinh(z)$, where $\kappa > 1$.

3. **Predator-prey models - linearization**. Let $x(t)$ denote the population of, say, rabbits on an island'at time t, and let $y(t)$ denote the population of foxes. The fox population will die out if there are not enough rabbits and it will grow in proportion to the number of rabbits: $y' = a(x - \mu)y$, where a and μ are positive constants. The rabbit population in the absence of foxes is assumed to grow logistically: $x' = rx(K - x)$, where the rate r and the carrying capacity K are positive numbers. However the presence of foxes will decrease the rabbit population in proportion to the size of both populations (the probability of a fox-rabbit encounter is proportional to xy). Hence we arrive at the following system:

$$\begin{aligned} x' &= rx(K - x) - bxy \\ y' &= a(x - \mu)y, \end{aligned}$$

where b is a positive constant. We assume that $K > \mu$. Then there is an equilibrium solution: $x = \mu$, and $y = \nu := r(K - \mu)/b$. We want to study the behavior of the populations near equilibrium, and so we make a change of variables:$\xi = x - \mu$ and $\eta = y - \nu$. After some algebra we see that

$$\begin{aligned} \xi' &= -r\mu\xi - b\mu\eta - r\xi^2 - b\xi\eta \\ \eta' &= a\nu\xi + a\xi\eta, \end{aligned}$$

Make another change of variables:$u = \xi/\mu$ and $v := \eta/nu$:

$$\begin{aligned} u' &= -r\mu u - b\nu v - r\mu u^2 - b\nu uv \\ v' &= a\mu u + a\mu uv, \end{aligned}$$

Assume that the populations hover close to equilibrium, i.e. ξ and η are small compared to the equilibrium populations: $u \ll 1$ and $v \ll 1$. On the premise that the terms involving u^2 and uv are negligible, we arrive at the *linearized* system:

$$\begin{aligned} u' &= -r\mu u - b\nu v \\ v' &= a\mu u, \end{aligned}$$

To investigate the behavior of the solution to this system we must find the eigenvalues of the coefficient matrix:

$$\lambda^2 + r\mu\lambda + ab\mu\nu = 0.$$

$$\lambda = \frac{-r\mu \pm \sqrt{r^2\mu^2 - 4ab\mu\nu}}{2}.$$

If $r^2\mu^2 > 4ab\mu\nu$ then both u and v will decrease exponentially to zero. If $r^2\mu^2 < 4ab\mu\nu$ then the trajectory $(u(t), v(t))$ will spiral towards the origin. If $r^2\mu^2 = 4ab\mu\nu$, the case of a repeated eigenvalue, then $u(t)$ and $v(t)$ will be made up of terms containing the factor $\mathrm{e}^{-r\mu t/2}$ and others containing $t\mathrm{e}^{-r\mu t/2}$. Both of these will converge to zero. Therefore, in all cases

$$\lim_{t\to\infty} x(t) = \mu, \quad \lim_{t\to\infty} y(t) = \nu.$$

Exercise. Show that if $K < \mu$ then

$$\lim_{t\to\infty} x(t) = K, \quad \lim_{t\to\infty} y(t) = 0.$$

In other words, the fox population will go extinct

Warning. This analysis was based on the assumption that the linearized equations really reflect the behavior near the equilibrium. There is in fact a theorem (the *Hartman-Grobman Theorem*) that says that this is true as long as the functions in the differential equation are smooth enough and as long as the linearized system has only eigenvalues with nonzero real part.

4. **Turtle survival - a linear system with three unknowns**. There have been several mathematical studies concerning sea turtles. A very simple example is the following. Let $x(t)$, $y(t)$ and $z(t)$ respectively represent the population of eggs, juveniles, and adults. Let a, b , c, k, ℓ, and m be positive constants representing the several transition rates. Then the following system of equations form a very rudimentary model:

$$\begin{aligned} x' &= az - kx \\ y' &= bx - ly \\ z' &= cy - mz \end{aligned}$$

Figure 3: Green sea turtle

It can be shown that if $k\ell m < abc$ then all solutions die exponentially (extinction of the population). To show this we note that the behavior of solutions is determined by the eigenvalues of the matrix

$$\mathbf{A} := \begin{pmatrix} -k & 0 & a \\ b & -\ell & 0 \\ 0 & c & -m \end{pmatrix}$$

whose characteristic equation is

$$\phi(\lambda) := -\det(\mathbf{A} - \lambda \mathbf{I}) = (k+\lambda)(\ell+\lambda)(m+\lambda) - abc = 0.$$

Note that $\phi(0) = k\ell m - abc$, and $\phi'(\lambda) > 0$ when $\lambda \geq 0$. Therefore, if $\phi(0) < 0$ there must be a positive root λ_0. Let $\hat{k} = k + \lambda_0$, $\hat{\ell} = \ell + \lambda_0$, and $\hat{m} = m + \lambda_0$. These are all positive numbers and the corresponding eigenvector $(u, v, w)^T$ must satisfy

$$\begin{pmatrix} -\hat{k} & 0 & a \\ b & -\hat{\ell} & 0 \\ 0 & c & -\hat{m} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

It is easily verified from these equations that if any of u, v, w are positive then all are positive and hence

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mathrm{e}^{\lambda_0 t} \tag{6}$$

represents a positive (i.e. physically realistic), exponentially growing solution. It is then possible to argue that with near certainty, any initial condition will lead to a solution that will be eventually dominated by a term such as given by (6), implying survival of the population. In other words, if $abc > k\ell m$ then the population will grow exponentially. If $abc < k\ell m$ then it can be shown (for those of you who know some linear algebra, the proof is given at the end of these notes) that all eigenvalues have negative real part and hence all solutions will die out exponentially.

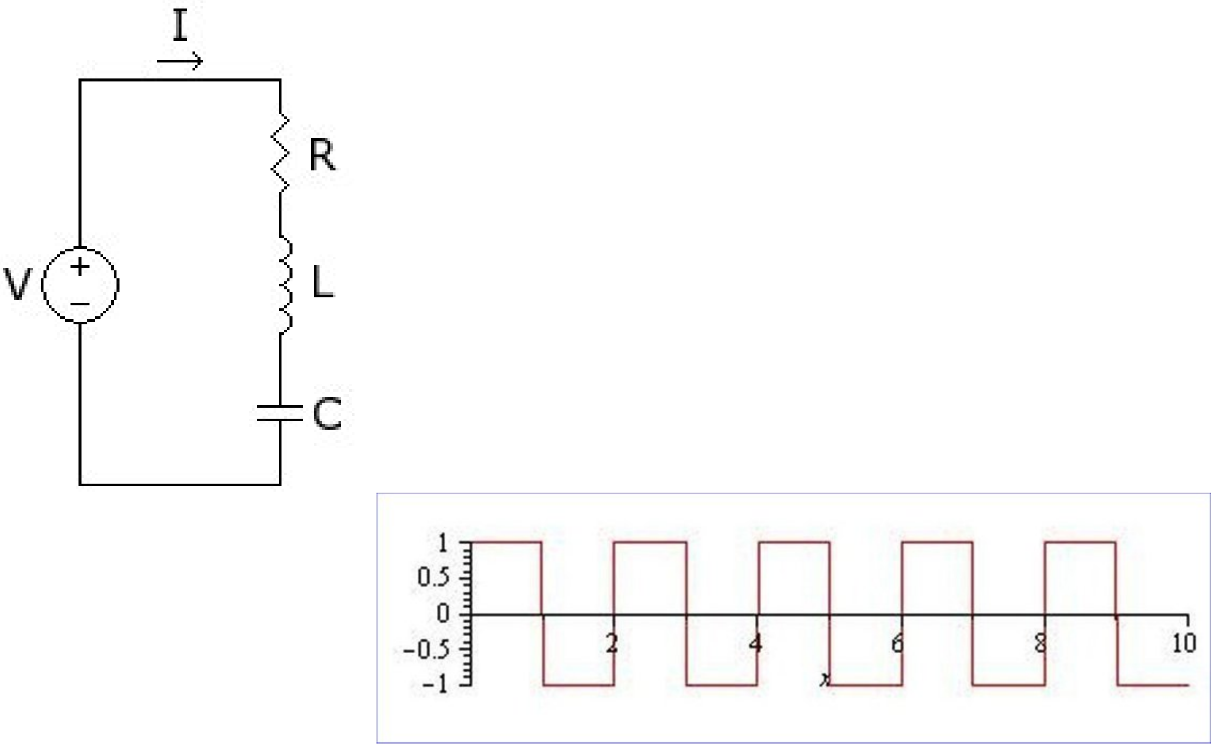

Figure 5: RLC circuit with square wave input

5. **RLC circuit with square wave input - Using the Laplace transform** .

Consider a simple RLC-circuit on which a voltage $V = E(t)$ is impressed. The differential equation for the current, $I = y(t)$ is

$$Ly'' + Ry' + y/C = E'(t),$$

where the positive numbers L, R, and C are respectively the inductance, resistance, and capacitance. Suppose that E is a sawtooth wave of period 1:

$$E(t) = \begin{cases} t \text{ if } 0 \le t < 1/2 \\ 1 - t \text{ if } 1/2 \le t < 1 \end{cases}$$

and $E(t + 1) = E(t)$ for all t. Let

$$f(t) = \begin{cases} 1 \text{ if } 0 \le t < 1/2 \\ -1 \text{ if } 1/2 \le t < 1 \end{cases}$$

Then

$$Ly'' + Ry' + y/C = f(t).$$

If assume initial conditions $y(0) = y'(0) = 0$ then

$$Y(s) = \frac{F(s)}{Ls^2 + Rs + s/C},$$

where

$$F(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt = \sum_{k=0}^{\infty} \int_k^{k+1} \mathrm{e}^{-st} f(t)\, dt.$$

In the k^{th} term make the change of variables $t = k + \tau$, then using the fact that $f(k + \tau) = f(\tau)$, we can compute:

$$F(s) = \sum_{k=0}^{\infty} \mathrm{e}^{-ks} \int_k^{k+1} \mathrm{e}^{-s\tau} f(\tau)\, d\tau = (1 - \mathrm{e}^{-s/2})^2 \sum_{k=0}^{\infty} \mathrm{e}^{-ks} = \frac{(1 - \mathrm{e}^{-s/2})^2}{s(1 - \mathrm{e}^{-s})}.$$

Therefore

$$Y(s) = \frac{(1-\mathrm{e}^{-s/2})^2}{s(1-\mathrm{e}^{-s})(Ls^2+Rs+s/c)}.$$

Writing $1-\mathrm{e}^{-s}$ as $(1-\mathrm{e}^{-s/2})(1+\mathrm{e}^{-s/2})$ this can be simplified to

$$Y(s) = \frac{(1-\mathrm{e}^{-s/2})}{s(1+\mathrm{e}^{-s/2})(Ls^2+Rs+s/c)}$$

Let's see if we can determine the behavior of the solution as $t \to \infty$. Using Taylor expansions:

$$Y(s) = \frac{1/2+\cdots}{L(1+\cdots)(s-\alpha_+)(s-\alpha_-)},$$

where α_+ and α_- are the two roots of Ls^2+Rs+s/C,

$$\alpha_\pm = \frac{-R \pm \sqrt{R^2-4L/C}}{2L}.$$

We see that there is no pole at the origin. Besides the poles α_+ and α_- that would occur in a solution to the homogeneous equation there are infinitely many others since $1+\mathrm{e}^{-s/2}=0$ whenever s is an odd multiple of $2\pi i$. We may therefore expect persistent oscillations.

Additional proofs.

Consider the matrix

$$\mathbf{A} := \begin{pmatrix} -k & 0 & a \\ b & -\ell & 0 \\ 0 & c & -m \end{pmatrix}.$$

The eigenvalues λ satisfy:

$$\phi(\lambda) := -\det(\mathbf{A}-\lambda\mathbf{I}) = (k+\lambda)(\ell+\lambda)(m+\lambda) - abc = 0.$$

Noting that $\phi'(\lambda) > 0$ whenever $\lambda \geq 0$ we have the following two cases:

1. If $\phi(0) = k\ell m - abc < 0$ then we have a positive root (eigenvalue) λ_0. If there are 3 real roots, then λ_0 is the largest eigenvalue and so the component of the solution to $\mathbf{u}' = \mathbf{A}\mathbf{u}$ corresponding to the eigenvalue λ_0 will grow and will eventually dominate. If the other two roots are complex, say $\alpha \pm i\beta$, then since the trace of $\mathbf{A}$, which is $-k-\ell-m$, is the sum of the eigenvalues, we see that $\alpha < 0$ and therefore, also in this case, the component of the solution corresponding to the eigenvalue λ_0 will grow and will eventually dominate.

2. If $\phi(0) = k\ell m - abc > 0$ then since $\phi'(\lambda) > 0$ for all $\lambda \geq 0$, we see that all real roots/eigenvalues are negative. The simplest case is that of 3 real eigenvalues. All of them must be negative and therefore all solutions of $\mathbf{u}' = \mathbf{A}\mathbf{u}$ die exponentially. To take care of the case of complex eigenvalues we need:

   Proposition. If λ_0 is a real eigenvalue of $\mathbf{A}$ then $\lambda_0 > -\max\{k,\ell,m\}$.
   Proof. Suppose that $\lambda_0 \leq -\max\{k,\ell,m\}$, then all the entries in the matrix $\mathbf{A}-\lambda_0\mathbf{I}$ are positive. It is then easy to see that if $(v_1,v_2,v_3)^T$ is an eigenvector then each entry v_i must have a different sign than any other entry, and that is obviously impossible.

   Since the trace of $\mathbf{A}$ is the sum of its eigenvalues, $2\alpha + \lambda_0 = -k-\ell-m$, so that $2\alpha < -k-\ell-m+\max\{k,\ell,m\} < 0$. Therefore components of a solution of $\mathbf{u}' = \mathbf{A}\mathbf{u}$ that correspond to these complex eigenvalues will also die out exponentially.