Structured SI Epidemic Models with Application in HIV Epidemics

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Outline

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1. Introduction on Epidemic Models
   - Demographical and Epidemiological Factors
   - Previous Work with ODE Models
2. Structured S-I Epidemic Models with Delay
   - Our Delay Model
   - Results and Simulations
3. Two Sex SI Epidemic Models
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4. Conclusions and Future Work
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5. Dedication and Acknowledgments

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Factors

With mathematical models, we can help determine who is at risk for disease and how changes to risk factors such as treatment and social behaviors can affect long-term health outcomes. For this purpose, we have to consider some basic factors as

- Demographical Factors : Sex (Female/Male), Age (Juvenile/Adult), Stage (Larvae/Adult)

Factors

With mathematical models, we can help determine who is at risk for disease and how changes to risk factors such as treatment and social behaviors can affect long-term health outcomes. For this purpose, we have to consider some basic factors as

- Demographical Factors : Sex (Female/Male), Age (Juvenile/Adult), Stage (Larvae/Adult)
- Epidemiological Factors : Susceptible, Infected, Recovered, in Quarantine, etc......

Outline

Classical SI Epidemic Models

$$\begin{aligned} \dot{S} &= b(N-S) - C_{SI}SI, \\ \dot{I} &= -\gamma I - bI + C_{SI}SI \ ; \end{aligned}$$

or

Characteristics

- Constant Total Population
- Constant Immigration Rate

Classical SI Epidemic Models

$$\begin{aligned}\dot{S} &= b(N-S) - C_{SI}SI,\\ \dot{I} &= -\gamma I - bI + C_{SI}SI\ ;\end{aligned}$$

or

$$\begin{aligned}\dot{S} &= U - \mu S - \frac{C_N SI}{S+I}\ ,\\ \dot{I} &= -(\gamma+\mu)I + \frac{C_N SI}{S+I}\ .\end{aligned}$$

Characteristics

- Constant Total Population
- Constant Immigration Rate

Stage structure and infectious disease (Thieme, H.)

Host parasite systems with stage-structured hosts
Only the larvae stage affected.

$$
\begin{aligned}
L &= S + I \ , \\
S' &= \beta(A)A - \mu(L)S - \gamma(L) - \eta SI \ , \\
I' &= \eta SI - (\mu(L) + \nu)I - q\gamma(L)I \ , \\
A' &= \gamma(L)S + pq\gamma(L)I - \alpha A \ .
\end{aligned}
$$

A Structured S-I Epidemic Model with two age (López, R.)

We set

J_1 : the size of the susceptible juvenile subpopulation ,
J_2 : the size of the infected juvenile subpopulation ,
A_1 : the size of the susceptible adult subpopulation ,
A_2 : the size of the infected adult subpopulation.

Assumptions

- Susceptible newborn is not necessarily of the same strain as their progenitor ,
- Only adults can reproduce (in consideration of the sex active individuals).

A Structured S-I Epidemic Model with two age (López, R.)

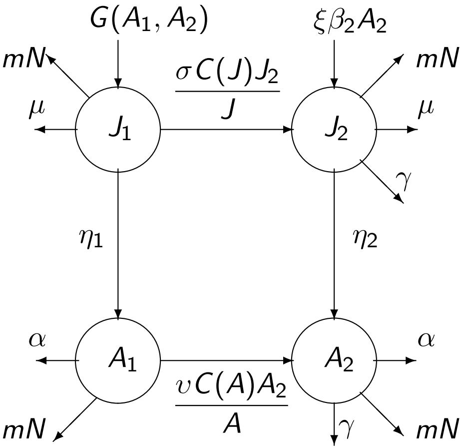

Figure: Flow diagram for the structured SI model with two age. Consider $G(A_1, A_2) = \beta_1 A_1 + (1-\xi)\beta_2 A_2$

A Structured S-I Epidemic Model with two age (López, R.)

$$
\begin{cases}
J_1' &= \beta_1 A_1 + (1-\xi)\beta_2 A_2 - \dfrac{\sigma C(J) J_1 J_2}{J} - \eta_1 J_1 - \mu J_1 - m J_1 N \ , \\
A_1' &= \eta_1 J_1 - \dfrac{\upsilon C(A) A_1 A_2}{A} - \alpha A_1 - m A_1 N \ , \\
J_2' &= \xi \beta_2 A_2 + \dfrac{\sigma C(J) J_1 J_2}{J} - \eta_2 J_2 - \mu J_2 - \gamma J_2 - m J_2 N \ , \\
A_2' &= \eta_2 J_2 + \dfrac{\upsilon C(A) A_1 A_2}{A} - \alpha A_2 - \gamma A_2 - m A_2 N \ ,
\end{cases}
$$

where $N(t) = J_1(t) + J_2(t) + A_1(t) + A_2(t)$.

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Considerations

The following integrals express the fact that the number of recruits into the juvenile (susceptible or infected) populations between times $t-(\theta+d\theta)$ and $t-\theta$ is given by $(\beta_1 A_1(t-\theta)+(1-\xi)\beta_2 A_2(t-\theta))d\theta$ for the case of susceptible population and $\xi\beta_2 A_2(t-\theta)d\theta$ in the case of infected one.

$$\begin{cases} J_1(t) = \int_0^{\tau} (\beta_1 A_1(t-\theta)e^{-\mu_1\theta} + (1-\xi)\beta_2 A_2(t-\theta)e^{-\mu_2\theta})d\theta \\ J_2(t) = \int_0^{\tau} \xi\beta_2 A_2(t-\theta)e^{-\mu_2\theta}d\theta \end{cases}$$

In consequence the delay model is given by

Delay Model

$$
\begin{cases}
J_1'(t) = \beta_1 A_1(t) + (1-\xi)\beta_2 A_2(t) - \mu_1 J_1(t) - \beta_1 A_1(t-\tau)e^{-\mu_1\tau} \\
\qquad -(1-\xi)\beta_2 A_2(t-\tau)e^{-\mu_2\tau} , \\
A_1'(t) = \beta_1 A_1(t-\tau)e^{-\mu_1\tau} + (1-\xi)\beta_2 A_2(t-\tau)e^{-\mu_2\tau} \\
\qquad -\dfrac{CA_1(t)A_2(t)}{A(t)} - \alpha A_1(t) , \\
J_2'(t) = \xi\beta_2 A_2(t) - \mu_2 J_2(t) - \xi\beta_2 A_2(t-\tau)e^{-\mu_2\tau} , \\
A_2'(t) = \xi\beta_2 A_2(t-\tau)e^{-\mu_2\tau} + \dfrac{CA_1(t)A_2(t)}{A(t)} - (\alpha+\gamma)A_2(t) ,
\end{cases}
$$

Delay Model

As we assume the solutions for $J_1(t)$ and $J_2(t)$, the delay model is reduced to solve

$$\begin{cases} A_1' = \beta_1 A_1(t-\tau)e^{-\mu_1\tau} + (1-\xi)\beta_2 A_2(t-\tau)e^{-\mu_2\tau} \\ \qquad -\dfrac{CA_1A_2}{A} - \alpha A_1 - mA_1A \ , \\ A_2' = \xi\beta_2 A_2(t-\tau)e^{-\mu_2\tau} + \dfrac{CA_1A_2}{A} - (\alpha+\gamma)A_2 - mA_2A \ , \end{cases}$$

Outline

Initial Conditions

The initial conditions are:

$$A_1(\theta) = \varphi_1(\theta) \geq 0\,, \qquad A_2(\theta) = \varphi_2(\theta) \geq 0\,, \qquad \theta \in [-\tau, 0]$$

Basic Results

Theorem

(Positivity) *The solutions of the system with the above initial conditions satisfy* $A_1(t) > 0$, $A_2(t) > 0$ *for all* $t > 0$.

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Theorem

(Boundedness) *For the given system, there exists an* $M > 0$, $A_1(t) \leq M$ *and* $A_2(t) \leq M$ *assuming* $0 \leq A_1(\theta) \leq M$, $0 \leq A_2(\theta) \leq M$, *for* $\theta \in [-\tau, 0]$.

Global Stability

First, we define the following threshold parameters

$$\mathbf{R}_1 = \frac{\beta_1 e^{-\mu_1 \tau}}{\alpha} \quad : \quad \text{average number of new susceptible juveniles produced by one typical susceptible adult},$$

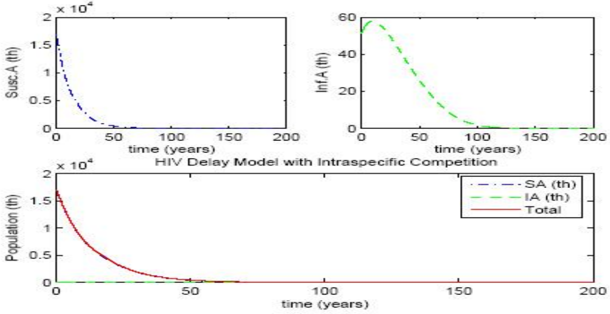
$$\mathbf{R}_2 = \frac{\beta_2 e^{-\mu_2 \tau}}{\alpha + \gamma} \quad : \quad \text{average number of new infected juveniles produced produced by one typical infected adult}.$$

Global Stability of Total Extinction Equilibrium

Theorem

Assume $\xi = 1$. *If* $\mathbf{R}_1, \mathbf{R}_2 < 1$, *then any solution of the given system satisfies* $\lim_{t \to \infty} (A_1(t), A_2(t)) = (0, 0)$.

Simulation: Global Stability Case $\mathbf{R}_1, \mathbf{R}_2 < 1$



Application: HIV in Peru (Estimated Parameters)

Parameter	Meaning	Value
β_1	Per Capita Birth Rate (SBR) of an average susceptible adult	0.0238 (yr^{-1})
β_2	Per Capita Birth Rate (IBR) of an average infected adult	0.00265 (yr^{-1})
μ	Per Capita Juvenile Death Rate (mean of JDR from 1992-2001)	0.0038 (yr^{-1})
α	Per Capita Death Rate (mean of CDR from 2001-2004)	0.01453 (yr^{-1})

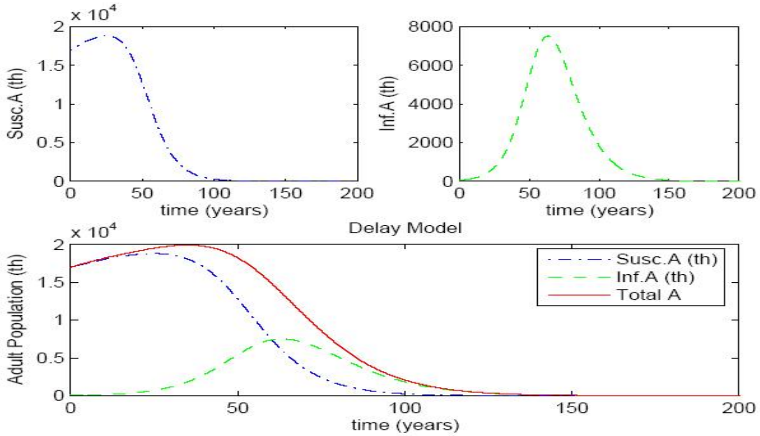
Table: All data was extracted from publications of peruvian and international organizations

Application: HIV in Peru (Estimated Parameters)

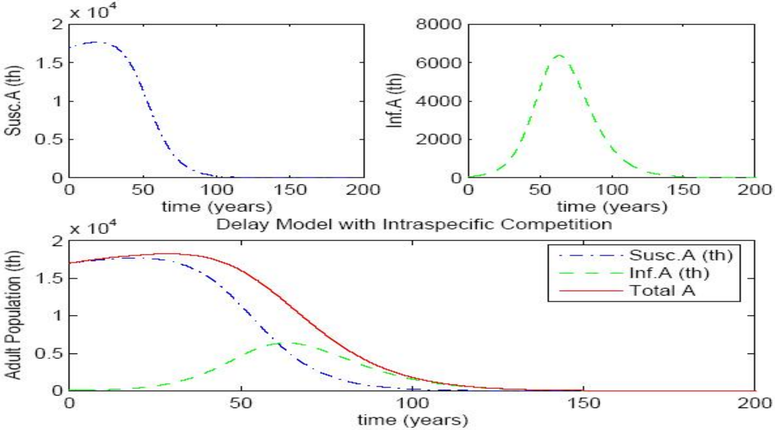
Parameter	Meaning	Value
γ	Per Capita HIV Related Death Rate (reciprocal of mean infection period)	0.072 (yr^{-1})
C	Per Capita Contact Rate (mean of CCR from 2001-2004)	0.1935 (yr^{-1})
τ	Time Delay (maturation period)	15 (yr)

Table: All data was extracted from publications of peruvian and international organizations

Simulation: HIV Peru Case: m=0



Simulation: HIV Peru Case: $m > 0$



Application: HIV in the USA (Estimated Parameters)

Parameter	Meaning	Value
β_1	Per Capita Birth Rate (SBR) of an average susceptible adult	0.01414 (yr^{-1})
β_2	Per Capita Birth Rate of an average infected adult	0.0015151851 (yr^{-1})
μ	Per Capita Juvenile Death Rate (mean of JDR from 1992-2001)	0.0038 (yr^{-1})
α	Per Capita Death Rate (mean of CDR from 2001-2004)	0.0128683567 (yr^{-1})

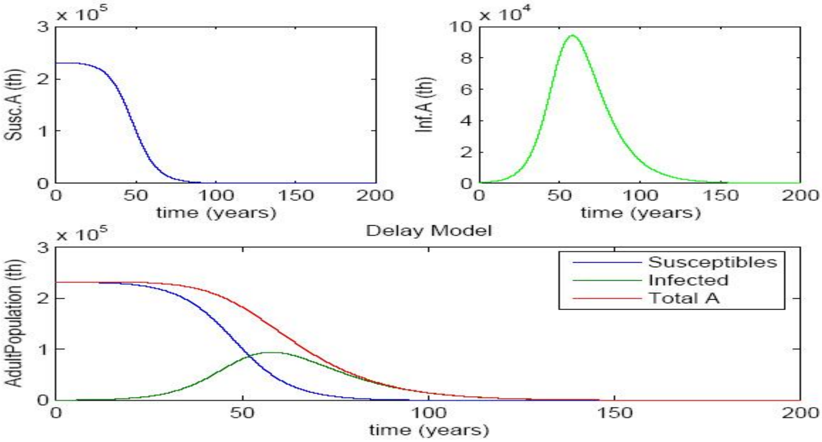
Table: All the data was extracted from american Organizations publications

Application: HIV in the USA (Estimated Parameters)

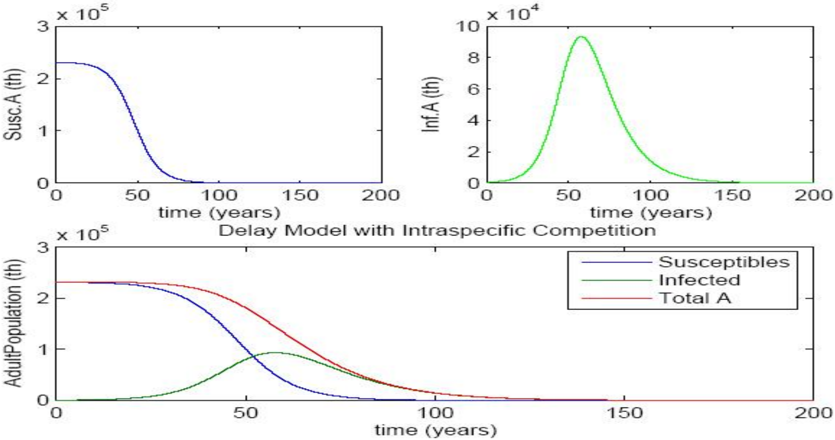
Parameter	Meaning	Value
γ	Per Capita HIV Related Death Rate (reciprocal of mean infection period)	0.0563115789 (yr^{-1})
C	Per Capita Contact Rate (mean of CDR from 2001-2004)	0.197 (yr^{-1})
τ	Time Delay (maturation period)	15 (yr)

Table: All the data was obtained from american Organizations publications

Simulation: HIV USA Case: $m = 0$



Simulation: HIV USA Case: $m > 0$



Outline

Flow Diagram

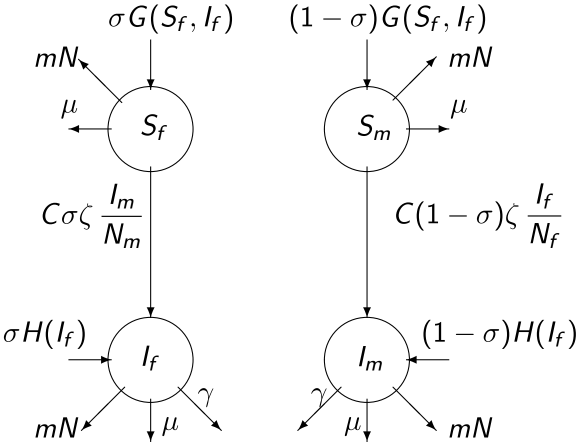


Figure: Flow diagram for the two-sex SI model, where $G(S_f, I_f) = \beta_1 S_f + (1-\xi)\beta_2 I_f$ and $H(I_f) = \xi\beta_2 I_f$

Model

$$
\begin{cases}
S_f' &= \sigma\beta_1 S_f + \sigma(1-\xi)\beta_2 I_f - C\sigma\zeta\dfrac{S_f I_m}{N_m} - \mu S_f - m S_f N \ , \\
S_m' &= (1-\sigma)\beta_1 S_f + (1-\sigma)(1-\xi)\beta_2 I_f - C(1-\sigma)\zeta\dfrac{S_m I_f}{N_f} \\
 & \quad -\mu S_m - m S_m N \ , \\
I_f' &= \sigma\xi\beta_2 I_f + C\sigma\zeta\dfrac{S_f I_m}{N_m} - (\mu+\gamma) I_f - m I_f N \ , \\
I_m' &= (1-\sigma)\xi\beta_2 I_f + C(1-\sigma)\zeta\dfrac{S_m I_f}{N_f} - (\mu+\gamma) I_m - m I_m N \ ,
\end{cases}
\tag{2}
$$

Outline

Basic Properties

Theorem

(Theorem of Existence, Uniqueness and Positivity) *For all* $S_f^0, S_m^0, I_f^0, I_m^0 > 0$, *there exists* $S_f, S_m, I_f, I_m : (0, \infty) \rightarrow (0, \infty)$ *which solves the given system and the initial conditions* $S_f = S_f^0, S_m = S_m^0, I_f = I_f^0, I_m = I_m^0$.

Basic Properties

Theorem

(**Theorem of Existence, Uniqueness and Positivity**) *For all $S_f^0, S_m^0, I_f^0, I_m^0 > 0$, there exists $S_f, S_m, I_f, I_m : (0, \infty) \to (0, \infty)$ which solves the given system and the initial conditions $S_f = S_f^0, S_m = S_m^0, I_f = I_f^0, I_m = I_m^0$.*

Theorem

(**Boundedness**) *All the solutions of the system are bounded.*

Thresholds Parameters

the average number of susceptible female newborn produced by a typical susceptible woman and

$$\tilde{\mathcal{R}}_1 = \frac{\sigma\beta_1}{\mu}$$

the average number of infected female newborn produced by a typical infected woman

$$\tilde{\mathcal{R}}_2 = \frac{\sigma\beta_2}{\mu+\gamma}$$

Steady States

Theorem

For the given system, the following statements are true
(a) If $\tilde{\mathcal{R}}_1 > 1$, then the system has a unique disease-free *equilibrium (DF) given by the expression*

$$DF = (\tilde{\mathcal{R}}_1 - 1)\left(\frac{\sigma\mu}{m}, \frac{(1-\sigma)\mu}{m}, 0, 0\right) \ .$$

Steady States

Theorem

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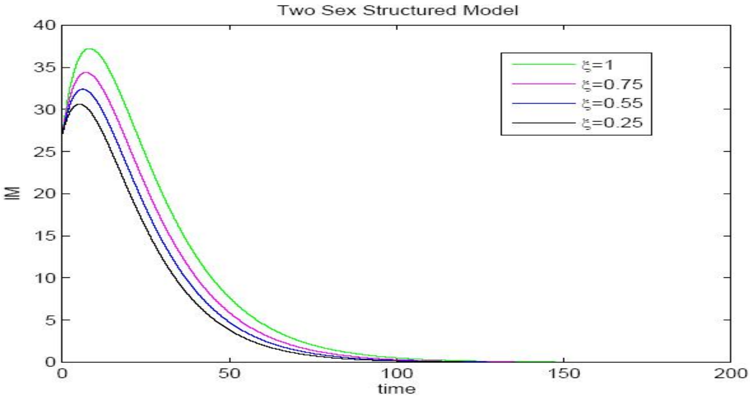
$$DF = (\tilde{\mathcal{R}}_1 - 1)\left(\frac{\sigma\mu}{m}, \frac{(1-\sigma)\mu}{m}, 0, 0\right) .$$

(b)If $\tilde{\mathcal{R}}_2 > 1$ and $\xi = 0$, then the system has a unique susceptible extinction *equilibrium (SE)given by the expression*

$$SE = (\tilde{\mathcal{R}}_2 - 1)\left(0, 0, \frac{\sigma(\mu+\gamma)}{m}, \frac{(1-\sigma)(\mu+\gamma)}{m}\right) .$$

Simulations: Two Sex SI Epidemic Models



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Conclusions

- The models accommodate a far more elaborate demographic consideration of the influence of age stages (juvenile and adult) on new infections.
- Sexual gender has also an influence in the study of sexually transmitted disease models.
- The simulations proved the importance of real data previous to the elaboration of more realistic models.

Outline

Future Work

- The concept of maturation process from juvenile to adult, depends on the conception of sexual activity beginning (changes in values of τ) It seems to be some oscillatory behavior due to these changes. Stability Switch
- The simulations of the Two sex model show the influence of the parameter of control on pregnant health. Then it will be useful to check the influence of immune system and drugs on infected mothers. Two-Sex SI Models including immunological influence

Dedication

In memory of my father
To my source of inspiration, my mother
To my brother

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