Sustained Oscillations via Coherence Resonance in SIR

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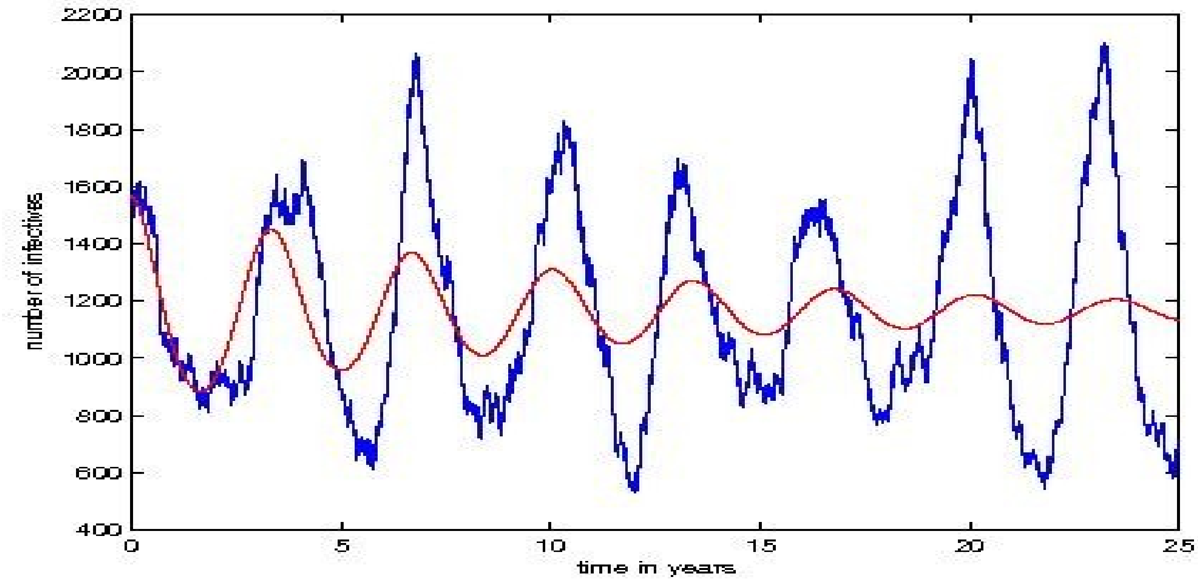
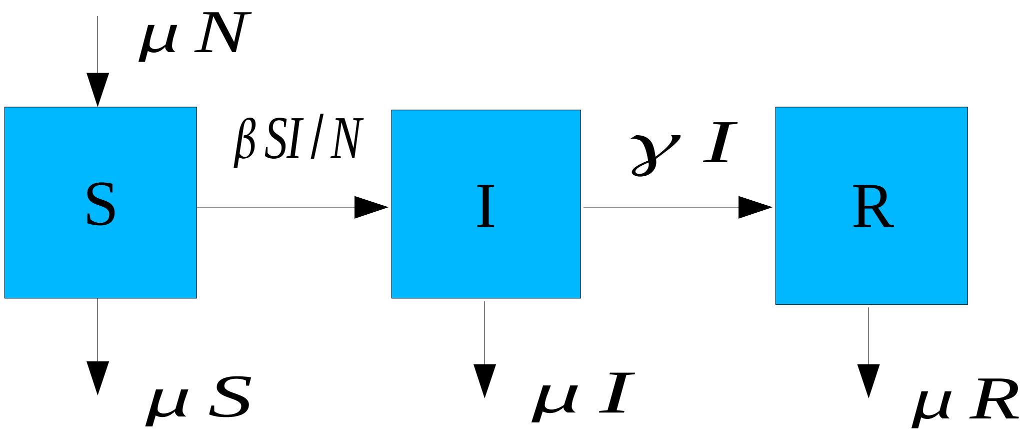


Example: influenza.

- Population = N = 2,000,000
- Average life span = $1/\mu$ = 80 years
- Reproductive number = R_0 = 15
- Time of infectivity = $1/\gamma$ = 15 days

The SIR model



$$\Delta S = (\mu(N-S) - \beta SI/N)\Delta t + \Delta Z_1 - \Delta Z_2$$

$$\Delta I = (\beta SI/N - (\gamma+\mu)I)\Delta t + \Delta Z_2 - \Delta Z_3$$

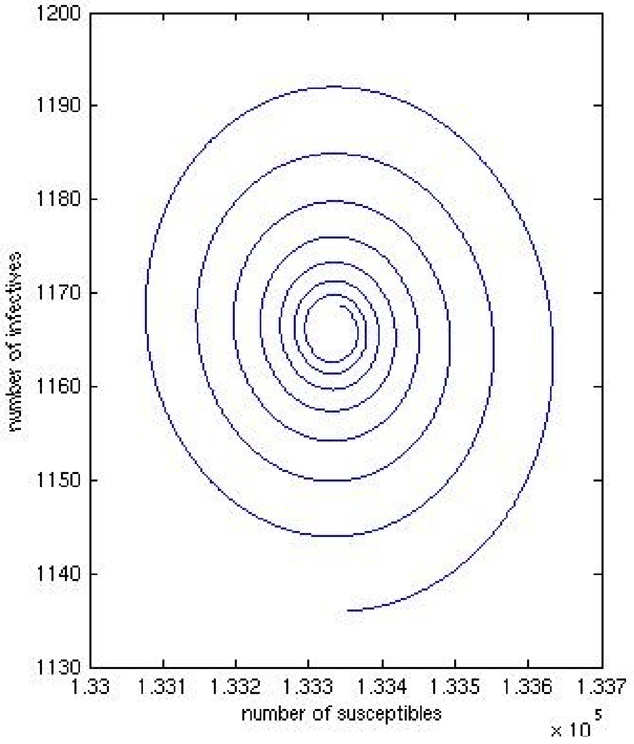
ΔZ_i *are centered Poisson increments with variances*

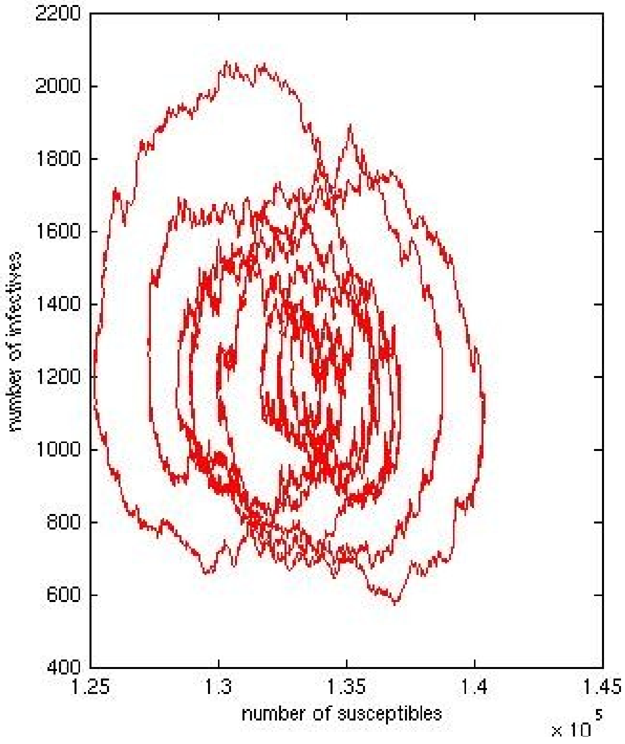
$$\mu(N+S)\Delta t,\ \beta SI/N\,\Delta t,\ (\gamma+\mu)I\,\Delta t.$$

Deterministic and stochastic paths in the phase plane, 25 years

The deterministic system has equilibrium at

$$S_{eq}=N/R_0, I_{eq}=N\mu(R_0-1)/\beta$$





Stochastic Differential Equations:

$$dS=(\mu(N-S)-\beta\frac{SI}{N})dt+\sqrt{\mu(N+S)}\,dW_1-\sqrt{\beta\frac{SI}{N}}\,dW_2$$

$$dI=(\beta\frac{SI}{N}-(\gamma+\mu)I)dt+\sqrt{\beta\frac{SI}{N}}\,dW_2-\sqrt{(\gamma+\mu)I}\,dW_3$$

Computations:

1. Change of variables:

$$u=\frac{S-S_{eq}}{S_{eq}} \qquad v=\frac{I-I_{eq}}{I_{eq}} \qquad \text{time:}\quad \Omega t,\ \Omega=\sqrt{\beta\frac{\mu}{R_0}(R_0-1)}$$

Now the stable equilibrium is at (0,0) and damped oscillations have frequency 1.

2. Linearize about (u,v)=(0,0), and set (u,v)=(0,0) in the noise coefficients

$$d\begin{pmatrix} u \\ v \end{pmatrix} = M\begin{pmatrix} u \\ v \end{pmatrix} dt + G\begin{pmatrix} dW_1 \\ dW_2 \\ dW_3 \end{pmatrix}.$$

The eigenvalues of M are $\lambda = -\epsilon^2 \pm \sqrt{\epsilon^4 - 1}$,

where $\epsilon^2 = \dfrac{\mu R_0}{2\Omega}$.

For $\epsilon \ll 1$, oscillations are slowly decaying with frequency 1.

Conjecture: The Stochastic model is approximately

$$\begin{pmatrix} u \\ v \end{pmatrix} = A(T)\begin{pmatrix} b\cos(t) \\ \sin(t) \end{pmatrix} + B(T)\begin{pmatrix} b\sin(t) \\ -\cos(t) \end{pmatrix}$$

where $b^2 = \frac{I_{eq}}{S_{eq}}, \quad T = \epsilon^2 t.$

A and B are diffusion processes with drift coefficients depending on (A,B) and constant diffusion coefficients.

Multiscale Analysis

Write (du,dv) using Ito's formula and compare with

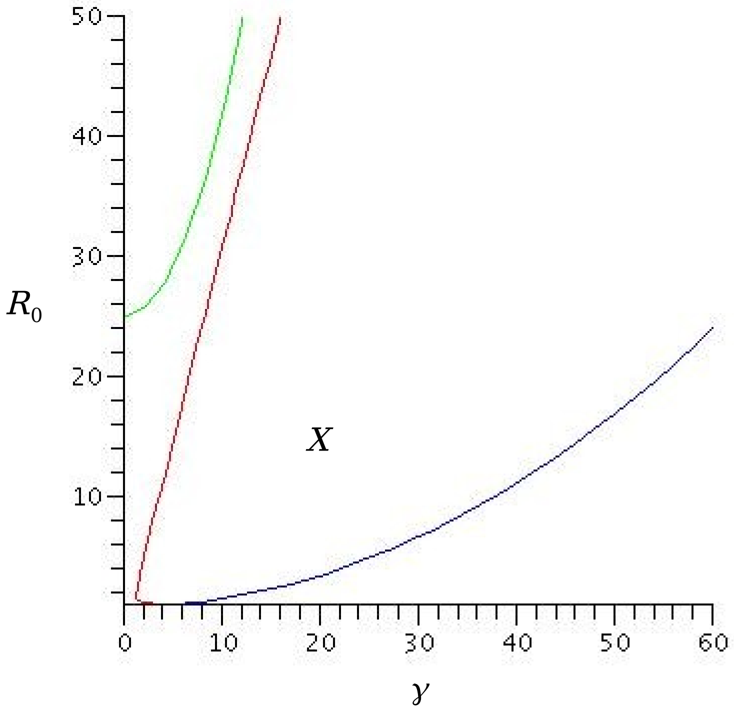
$$d\begin{pmatrix} u \\ v \end{pmatrix} = M\begin{pmatrix} u \\ v \end{pmatrix} dt + G\begin{pmatrix} dW_1 \\ dW_2 \\ dW_3 \end{pmatrix}.$$

1. Equate drift and diffusion coefficients. Obtain 4 equations.
2. Integrate each with respect to t over [0,2π].
3. Consider functions of T as constant in t over [0,2π].
4. Identify the drift and diffusion coefficients in the SDE of A(T) and B(T):

$$d\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} dT + \frac{\delta}{\sqrt{2\epsilon}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} dW_1 \\ dW_2 \end{pmatrix}.$$

(A,B) are independent Ornstein-Uhlenbeck processes.

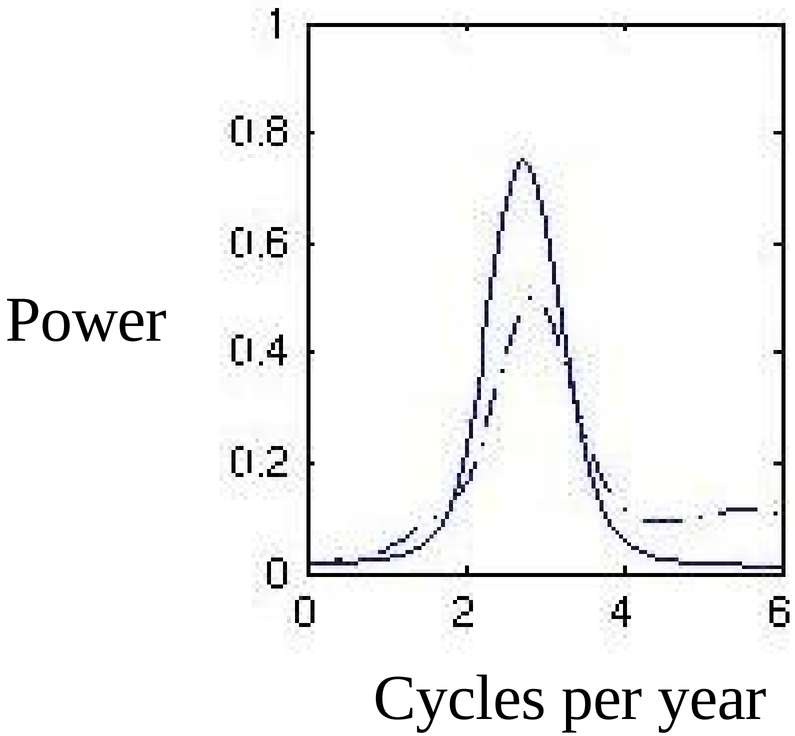
Parameter region for Ro and γ



- We need $\epsilon^2 << 1$ for separation of scales; $\delta^2/2\epsilon^2$, the stationary variance of A and B, of moderate size.
- Red: $\epsilon^2 = 0.1$
- Blue: $\delta^2/2\epsilon^2 = 1$
- Green: $\delta^2/2\epsilon^2 = 0.04$
- N = 500,000, $\mu = 1/55$
- X = (25,15)

Power Spectral Density of Stochastic Model and Multiscale Approximation

- Solid: Approximation
- Dot-dash: Stochastic Model
- Ro = 15, γ = 25
- N = 500,000, μ = 1/55



Histograms produced with 100 realizations during 200 years