

(1) Consider the following simple SI model:

$$\begin{cases} x'(t) = x + y - (1 - \theta)y - x(x + y) - \delta x - s\frac{xy}{x + y} \equiv F(x, y), x(0) = x_0 > 0 \\ y'(t) = -(\delta + r)y - y(x + y) + s\frac{xy}{x + y} \equiv G(x, y), y(0) = y_0 > 0 \end{cases} \quad (1)$$

where s, δ, r are positive constants and $\theta \in [0, 1]$. To remove the nonsmoothness at the origin, we carry out the blow-up transformation $(x, y) \rightarrow (u, y)$ where $u = x/y$ in system (1). This transforms it into the following system:

$$\begin{cases} u'(t) = (1 + r - s)u + \theta, \\ y'(t) = \varphi(u)y - (u + 1)y^2, \end{cases} \quad (2)$$

where

$$\varphi(u) = s - r - \delta - \frac{s}{u + 1}, \quad (3)$$

i): Let $(x(t), y(t))$ be the solution of (1), starting at $(x_0, y_0) \in \mathbf{R}_+^2$. Then there is a $t_0 > 0$ such that for $t > t_0$, we have $(x(t), y(t)) \in \Omega = \{(x, y) \in \mathbf{R}_+^2 \mid x + y < 1\}$.

ii): The equilibrium $(1 - \delta, 0)$ is globally attractive if either $s - r < 1$ and $\theta \in [0, 1]$,

or $s - r = 1$ and $\theta \in (0, 1]$ hold.

iii): The equilibrium $(0, 0)$ is globally attractive if $0 \leq \theta \leq 1 < s - r$ and $\varphi(u^*) \leq 0$ where $u^* = \frac{\theta}{s - r - 1}$.

(Reference: T.-W. Hwang and Y. Kuang: Host extinction dynamics in a simple parasite-host interaction model, *Math. Biosc. and Eng.*, **2**, 743-751, (2005))

(2) Consider

$$\begin{cases} S' = -SI + 2(4 - S - I), S(0) > 0 \\ I' = SI - 2I, I(0) > 0. \end{cases}$$

a): Show that solutions with positive initial data are positive and bounded and

$$\limsup_{t \rightarrow \infty} (S(t) + I(t)) \leq 4$$

if $S(0) + I(0) < 4$.

b): Show that the region $\Delta = \{(S, I) : 0 < S + I < 4, S > 0, I > 0\}$ is invariant.

c): Show that this SI epidemic model has no limit cycle in the positive cone and the positive steady state $(2, 1)$ is globally asymptotically stable with respect to positive initial data. (Hint: use Dulac criterion).

(3) Consider the following food chain

$$\begin{cases} x' = x(5 - 2x - y), \\ y' = y(-1 + 3x - z - y), \\ z' = z(-1 + y). \end{cases}$$

a): Find the local stability of the positive steady state of the food chain by the Routh–Hurwitz criteria.

b): Show that the positive steady state is globally stable.

(4): Exercise 21, page 206.

(5): Exercise 24, page 208.

(6): Exercise 22, page 261.

- (7): Exercise 30, page 264.
- (8): Exercise 33, page 266.
- (9): Exercise 35, page 267.