

## Lectures 8, Th., Sept. 13

### Reading homework: Chapter 2

**1. Lyapunov exponent.** Adapted from Wikipedia, the free encyclopedia.

The Lyapunov exponent or Lyapunov characteristic exponent of a dynamical system, say  $x_{n+1} = f(x_n)$ ,  $f \in C^1$  is a quantity that characterizes the rate of separation of infinitesimally close trajectories. Quantitatively, two trajectories in phase space (starting at  $x_0$  and  $y_0$ ) with initial separation  $\delta_0 = y_0 - x_0$  diverge

$$\delta_n \equiv y_n - x_n = f^n(y_0) - f^n(x_0) = \delta_0 e^{\lambda_n n}.$$

Hence

$$\lambda_n = \frac{1}{n} \ln\left(\frac{\delta_n}{\delta_0}\right) = \frac{1}{n} \ln\left(\frac{f^n(y_0) - f^n(x_0)}{y_0 - x_0}\right).$$

When  $\delta_0$  is very small, we can approximate  $\lambda_n$  by

$$\lambda_n \approx \frac{1}{n} \ln(f'^n(x_0)).$$

Notice that

$$f'^{(n)}(x_0) = \prod_{i=0}^{n-1} f'(x_i).$$

We have

$$\lambda_n \approx \frac{1}{n} \ln f'^{(n)}(x_0) = \frac{1}{n} \sum_{i=0}^{n-1} \ln f'(x_i).$$

We call the limit of  $\lambda_n$  as  $n \rightarrow \infty$ , when it exists, a Lyapunov exponent.

For a dynamical system of dimension larger than 1, the rate of separation can be different for different orientations of initial separation vector. Thus, there is a whole spectrum of Lyapunov exponents—the number of them is equal to the number of dimensions of the phase space. It is common to just refer to the largest one as the Lyapunov exponent, because it determines the predictability of a dynamical system.

#### Basic properties

If the system is conservative (i.e. there is no dissipation), a volume element of the phase space will stay the same along a trajectory. Thus the sum of all Lyapunov exponents must be zero. If the system is dissipative, the sum of Lyapunov exponents is negative.

If the system is a flow, one exponent is always zero—the Lyapunov exponent corresponding to the eigenvalue of  $L$  with an eigenvector in the direction of the flow.

The inverse of the largest Lyapunov exponent is sometimes referred to in literature as Lyapunov time. For chaotic orbits, the Lyapunov time will be finite, whereas for regular orbits it will be infinite.

**Local Lyapunov exponent:** Whereas the (global) Lyapunov exponent gives a measure for the total predictability of a system, it is sometimes interesting to estimate the local predictability around a point  $x_0$  in phase space. This may be done through the eigenvalues of the Jacobian matrix  $J_0(x_0)$ . These eigenvalues are also called local Lyapunov exponents. The eigenvectors of the Jacobian matrix point in the direction of the stable and unstable manifolds.

**2. Tent map.** Adapted from Wikipedia, the free encyclopedia

In mathematics, the tent map is an iterated function, in the shape of a tent, forming a discrete-time dynamical system.

$$x_{n+1} = \begin{cases} \mu x_n & \text{for } x_n < \frac{1}{2} \\ \mu(1 - x_n) & \text{for } \frac{1}{2} \leq x_n, \end{cases}$$

where  $\mu$  is a positive real constant.

**Behavior**

The tent map and the logistic map are topologically conjugate, and thus the behavior of the two maps are in this sense identical under iteration.

Depending on the value of  $\mu$ , the tent map demonstrates a range of dynamical behavior ranging from predictable to chaotic. If  $\mu$  is less than 1 the point  $x = 0$  is a globally attractive fixed point of the system. If  $\mu$  is 1, all values of  $x$  less than or equal to  $1/2$  are fixed points of the system. If  $\mu$  is greater than 1 the system has two fixed points, one at 0, and the other at  $\mu/(\mu + 1)$ . Both fixed points are unstable.

**3. Stability of second order nonlinear difference equations.** Read sections 2.7 and 2.8. Make sure that you understand and familiar with the following result.

THEOREM 1. *The roots of  $\lambda^2 - \beta\lambda + \gamma = 0$  satisfy  $|\lambda| < 1$  if  $|\beta| < 1 + \gamma < 2$ .*

**4. Stability of higher order nonlinear difference equations.** Read section 2.9. Make sure that you are familiar with Jury test in the special case of third order difference equations.

THEOREM 2. *The roots of  $P(\lambda) \equiv \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$  satisfy  $|\lambda| < 1$  if and only if that*

1):  $P(1) > 0$ ,

2):  $P(-1) < 0$ ,

3):  $|a_3| < 1, |b_3| > |b_1|, |c_3| > |c_2|$ ,

where  $b_3 = 1 - a_3^2, b_2 = a_1 - a_3a_2, b_1 = a_2 - a_3a_1, c_3 = b_3^2 - b_1^2, c_2 = b_3b_2 - b_1b_2$ .

The Jury test is a practical presentation of the so-called Schur-Cohn criterion which can be derived easily from the much more well-known Routh-Hurwitz criterion (see page 233).