Lectures 7, Tu., Sept. 11

Reading homework: Chapter 2

1. Some global stability results and their applications to logistic difference equation. In this section we shall present some additional general yet effective results on global asymptotical stability of certain scalar nonlinear difference equations. Specifically, we are concerned here

$$x_{n+1} = f(x_n), \qquad x_0 = a, \tag{1.1}$$

where $f(x)$ is continuous and monotone or has a single peak(hump) in the domain of interest. Such functions are often encountered in applications. Some of the results of this section will be applied to specific equations in subsequent sections. We shall assume that the equation is defined on an interval of the real line. Without lose of generality, we shall further assume that such interval is invariant for the considered equation. Such assumptions are often readily met in applications. In what follows, we denote by x_n or $x_n(a)$ the solution of (1.1).

In applications, we often encounter functions of f that has a single peak. In fact, it frequently satisfies the following property:

(i): There exists $x_M \in I$, such that $f'(x)(x - x_M) < 0$ for $x \neq x_M, x \in I$.

Clearly, x_M is the peak. The next two theorems deals with such functions. The next theorem considers the case that the peak is located at the steady state or on the right hand side of the steady state.

Theorem 1.1. *Let I be an interval of real line. Assume that I is invariant with respect to* (1.1) *and f satisfies property (i). If* (1.1) *has a unique steady state $x^* \in I$, such that $x^* \leq x_M$. Then this steady state is globally asymptotically stable on I.*

Proof. From the assumptions about the function $f(x)$ and the uniqueness of the steady state x^*, we see that (1): if $x^* > x_0 \in I$, then $\{x_n\}_0^\infty$ is an increasing sequence satisfying $\lim_{n\to\infty} x_n = x^*$. If $x^* = x_M$, then it is easy to see that (2): if $x_0 > x^*$, then $x_1 < x^*$. This reduces this case to case (1), and therefore, we see that x_n tends to x^* as n tends to infinity. Assume below that $x_M > x^*$. Similarly, it is easy to see that (3): if $x_0 \in [x^*, x_M]$, then $\{x_n\}_0^\infty$ is decreasing with limit x^*. Let

$$x_* = \max\{x : x \in I, f(x) = x^*\}.$$

If $x_* = x^*$, then for $x_0 > x_M$, we must have $x_1 \in (x^*, x_M)$. Hence, by (3), we see that $x_n, n > 0$ form a monotone decreasing sequence with limit again x^*. If $x_* \neq x^*$. Then for $x_0 \in [x_M, x_*]$, we see as before that $x_1 \in [x^*, x_M]$, and hence by (3), we see that $\{x_n\}_1^\infty$ is decreasing with limit x^*. If $x_0 > x_*$, we have $x_1 < x^*$, hence by (1), we obtain an increasing sequence $\{x_n\}_1^\infty$ with limit x^*. This proves the theorem. ∎

We shall show below that if the positive steady state $x^* = 1 - r^{-1}$ of the logistic difference equation

$$x_{n+1} = r x_n (1 - x_n), \qquad x_0 \in [0, 1] \tag{1.2}$$

is locally asymptotically stable(when $1 < r < 3$), then it is in fact is globally asymptotically stable with respect to positive solutions. Indeed, we shall show that it is globally asymptotically stable even when $r = 3$. We will divide this task into two steps. In the first step, we apply Theorem 1.1 to show that the above statement is true if $1 < r \leq 2$.

LEMMA 1.2. *Assume that* $1 < r \leq 2$. *Then the steady state* $1 - r^{-1}$ *of the logistic difference equation* (1.2) *is globally asymptotically stable with respect to positive solutions with* $x_0 \in (0, 1)$.

Proof. We note that the peak of $f(x) = rx(1-x)$ is at $x = 1/2$ and $f(1/2) = r/4 \leq 1/2$. Let $I = (0, 1)$, then we see that I is invariant for (1.2). Clearly, f satisfies the property (i)(see the previous section) and the peak is located at the steady state or on the right hand side of it. The conclusion of the lemma follows from that of Theorem 1.1. ∎

Our next theorem deals with the case when the peak is on the left side of the steady state. We need the following notation.

$$x_l = \min\{x : x \in I, f(x) = x^*\}, \qquad b = f(x_M), \qquad a = f(b).$$

We note that $a < x^* < b$ and if $x_l \neq x^*$, then $x_l < x_M < x^*$. We also need the following observation.

LEMMA 1.3. *Let* I *be an interval of real line. Assume that* I *is invariant with respect to* (1.1) *and* f *satisfies property (i). If* (1.1) *has a unique steady state* $x^* \in I$, *such that* $x^* > x_M$. *Then for any* $x_0 \in I$, *there is a nonnegative integer* $m = m(x_0)$, *and* $c = c(x_0), c \in [x_M, x^*]$ *such that* $f(x_m) = f(c)$. *Moreover,* $[a, b]$ *is invariant for* (1.1).

Proof. Observe first that if $x_0 > x^*$, then $x_1 < x^*$. So, we may assume that $x_0 < x^*$. If $x_l \neq x^*$, then we see there is a nonnegative integer m such that $x_m \in [x_l, x^*]$. It is clear that there is a unique $c \in [x_M, x^*]$ such that $f(x_m) = f(c)$. Also, we see that $x_{m+1} \in [x^*, b]$ and $x_{m+2} \in [a, x^*]$. Indeed, for all positive integer $n > m$, we have $x_n \in [a, b]$. If $x_l = x^*$, then it is also obvious that there is a unique $c \in [x_M, x^*]$ such that $f(x_0) = f(c)$. In this case, $x_1 \in [x^*, b]$ and $x_2 \in [a, x^*]$. The invariance of $[a, b]$ is obvious. This proves the lemma. ∎

Note that the next theorem gives both necessary and sufficient conditions for the global stability of a steady state.

THEOREM 1.4. *Let* I *be an interval of real line. Assume that* I *is invariant with respect to* (1.1) *and* f *satisfies property (i). Assume also that* (1.1) *has a unique steady state* $x^* \in I$, *such that* $x^* > x_M$. *Then* x^* *is globally asymptotically stable on* I *if and only if that* $f(f(x)) > x$ *for* $x \in [x_M, x^*)$.

Proof. It is easy to see that if $f(f(x_p)) = x_p$ for some $x_p \in [x_M, x^*)$, then $x_p, f(x_p)$ form a period two cycle. If $f(f(x)) < x$ for all $x \in [x_M, x^*)$, then we see that x^* is unstable.

From the previous lemma, we see that for any $x_0 \in I$, there exists nonnegative integer m, such that $f(x_m) = f(c)$ for some $c \in [x_M, x^*]$. So, without loss of generality, we assume below that $m = 0$. If $f(f(x)) > x$ for $x \in [x_M, x^*)$, then we see that x_{2n} is strictly increasing and bounded above by x^*. Let

$$\alpha = \lim_{n \to i} x_{2n}.$$

Then we have

$$f(f(\alpha)) = \lim_{n \to i} x_{2n+2} = \alpha.$$

Therefore, we must have $\alpha = x^*$. This also yields

$$\lim_{n\to i} x_{2n+1} = f(x^*) = x^*.$$

This concludes the proof. ∎

It is not difficult to see from the proofs of the above theorems that if function f does not satisfies property (i), but has a unique steady state x^* in I, then we still have the following result.

THEOREM 1.5. *Let I be an interval of real line. Assume that I is invariant with respect to* (1.1) *and that* (1.1) *has a unique steady state $x^* \in I$. Let*

$$x_M = \sup\{x : x \in I, x \leq x^*, f(x) \geq f(s), s \leq x^*\}.$$

(a):If $x_M = x^*$, then x^* is globally asymptotically stable on I
(b): If $x^* > x_M$, then x^* is globally asymptotically stable on I if and only if that $f(f(x)) > x$ for $x \in [x_M, x^*)$.

We now apply Theorem 1.4to the logistic difference equation.

LEMMA 1.6. *Assume that $2 < r \leq 3$. Then the steady state $1 - r^{-1}$ of the logistic difference equation* (1.2) *is globally asymptotically stable with respect to positive solutions with $x_0 \in (0, 1)$.*

Proof. In this case, we note that the peak $x = 1/2$ of $f(x) = rx(1-x)$ is on the left hand side of the steady state $x^* = 1 - r^{-1}$. Again we have $I = (0, 1)$ and f satisfies the property (i). The conclusion of the theorem follows from that of theorem 1.5, if we can show that $f(f(x)) > x$ for $x \in [1/2, 1 - r^{-1})$. This is equivalent to show that

$$r^2 x(1-x)[1 - rx(1-x)] > x,$$

or

$$G(x, r) \equiv r^3 x(1-x)^2 - r^2(1-x) + 1 < 0,$$

for

$$(x, r) \in D \equiv [1/2, 1 - r^{-1}) \times (2, 3].$$

Notice that $G(1 - r^{-1}, r) = 0$, and

$$G(1/2, r) \equiv h(r) = r^3/8 - r^2/2 + 1.$$

Note that $h(2) = 0, h(3) = -1/8 < 0$, and $h'(r) = r(3r/8 - 1)$. This indicates that h first decreases strictly on $(2, 8/3)$, followed by strictly increasing on $[8/3, 3]$. Clearly, this shows that $h(r) < 0$, for $r \in (2, 3]$. Notice also that

$$\frac{\partial G}{\partial x} \equiv G_x(x, r) = r^3(1-x)^2 - 2r^3 x(1-x) + r^2 = r^2[3r(x - 2/3)^2 + 1 - r/3].$$

Clearly, $G_x(x, r) \geq 0$ for $2 < r \leq 3$. $G_x(x, r) = 0$ if and only if $r = 3, x = 2/3$. However the point $(x, r) = (2/3, 3)$ is not inside the set D of consideration. This proves that $G_x(x, r) > 0$ in the set of interest D. Hence, we must have for $(x, r) \in D$,

$$G(x, r) < 0.$$

This completes the proof. ∎

Clearly, Lemma 1.2 and Lemma 1.6 together imply the following theorem.

THEOREM 1.7. *If $1 < r \leq 3$, then the positive steady state $1 - r^{-1}$ of the logistic difference equation* (1.2) *is globally asymptotically stable with respect to positive solutions with $x_0 \in (0, 1)$.*