

A MECHANISTIC DERIVATION OF THE LOGISTIC MODEL

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(Adapted from Kuang et al. 2004)

ABSTRACT. The purpose of this note is to mechanistically derive the logistic population growth models from the well tested and received Droop equation.

1. A mechanistic derivation of the logistic equation. Population growth involves and often is determined by the birth and death processes. Most of the existing studies focus on birth process on the combined birth and death processes (the growth process). In general, death mechanisms are more numerous and difficult to study than birth mechanisms in a lab or field setting. In a short time frame, growth dynamics can be approximated by a linear differential equation with the coefficient called growth rate. Longer term, this growth rate shall be regarded as time dependent, or density dependent. The so-called Droop equation provides a time and experiment tested simple mathematical expression for biomass growth rate. We shall show that it also provides a convenient base for deriving the classical logistic equation. This section is adapted from Kuang et al. (2004).

In 1968, Droop reported some surprising findings based on his most ambitious and comprehensive chemostat experiment to date in terms of concept, technical difficulty and mathematical analysis, it was to surpass by far all that had gone before (Leadbeater, 2006). The experiment studied the kinetics of vitamin B₁₂ limitation in *Monochrysis lutheri* in continuous and exponentially growing batch cultures and in washed cell suspensions. The aim of this experiments was to relate specific growth rate to substrate concentration (Droop 1968). Contrary to conventional belief, the specific growth rate (μ) of *Monochrysis* in the chemostats was found not to depend directly on medium substrate concentration. However, the one relationship that did stand out was that growth depended on the intracellular concentration of vitamin B₁₂ (cell quota Q). The relationship between specific growth rate (μ) and cell quota (Q) took the following simple form (Droop 1973, 1974).

$$\mu = \mu_m \left(1 - \frac{q}{Q}\right). \quad (1.1)$$

This equation is called the Droop equation. We call $\mu_m \left(1 - \frac{q}{Q}\right)$ the Droop function. The parameter q is the minimum quota necessary for life (the subsistence quota) and represents the value of cell quota Q at zero growth rate. μ_m is the growth rate

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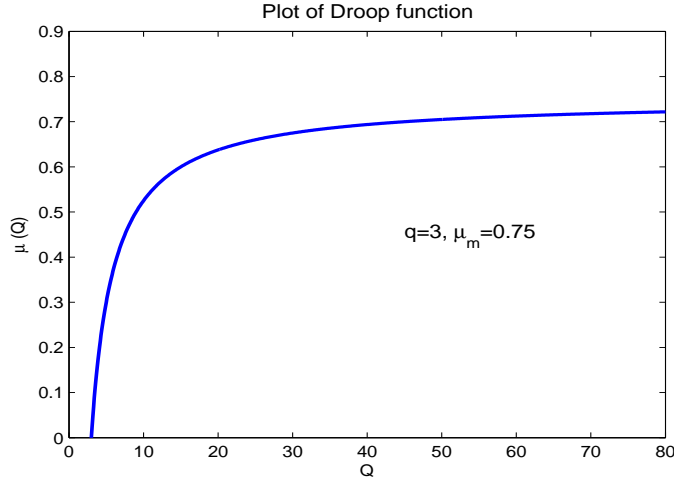


FIGURE 1. Plot of Droop function $\mu_m \left(1 - \frac{q}{Q}\right)$.

at infinite internal nutrient content which is clearly unattainable in experiments. To a certain degree, Droop equation

Our main purpose in this lecture is to derive the logistic equation via Droop equation. We consider a single species growing in a closed environment where there is a single limiting nutrient. For convenience, we assume below this limiting nutrient is phosphorus P . Hence the total amount of phosphorus P_t in the environment remains constant.

If we let P_x and P_f be the phosphorus in the species and the free phosphorus respectively, then $P_t = P_x + P_f$. Let $x = x(t)$ be the species density and $Q = Q(t)$ be the species' cell quota for P . Then $P_x = Qx$. Hence

$$P_t = P_f + Qx. \quad (1.2)$$

In the following, we let q be the species' minimal cell quota for P , μ_m be the species' maximal growth rate, D be its death rate. By (1.1), we have the following equation for the species growth:

$$\frac{dx}{dt} = \mu_m \left(1 - \frac{q}{Q}\right)x - Dx. \quad (1.3)$$

We need an equation governing the dynamics of Q , the species' cell quota for P . We assume that Q 's recruitment comes proportionally from the free phosphorus (αP_f) and its depletion because of cell growth is $\mu_m(Q - q)$. This results in the following simple equation

$$\frac{dQ}{dt} = \alpha P_f - \mu_m(Q - q). \quad (1.4)$$

We assume that $Q(0) \geq q$. Mathematically, this ensures that $Q(t) \geq q$ for all $t > 0$.

Since the cell metabolic process operates at a much faster pace than the growth of total biomass of a species, the quasi-steady-state argument allows us to approximate $Q(t)$ by the solution of

$$\alpha P_f - \mu_m(Q - q) = 0, \quad (1.5)$$

which takes the form of

$$Q = \frac{\alpha P_f + q\mu_m}{\mu_m}. \quad (1.6)$$

This together with (1.2) yields

$$P_f = \frac{\mu_m}{\mu_m + \alpha x} (P_t - qx). \quad (1.7)$$

Substituting (1.7) into (1.6) yields

$$Q = q + \frac{\alpha}{\mu_m + \alpha x} (P_t - qx). \quad (1.8)$$

Substituting the above into (1.3) and applying some straightforward simplification yields

$$\frac{dx}{dt} = \mu_m x \frac{P_t - qx}{P_t + \mu_m q \alpha^{-1}} - Dx. \quad (1.9)$$

The above equation can be rewritten as

$$\frac{dx}{dt} = \mu_m x \left(1 - \frac{qx + \mu_m q \alpha^{-1}}{P_t + \mu_m q \alpha^{-1}} \right) - Dx. \quad (1.10)$$

We can rewrite the above equation as

$$\frac{dx}{dt} = (\mu_m - D)x \left[1 - \frac{x + \mu_m \alpha^{-1}}{[(\mu_m - D)/\mu_m][\mu_m \alpha^{-1} + P_t/q]} \right]. \quad (1.11)$$

Or equivalently,

$$\frac{dx}{dt} = \frac{(\mu_m - D)P_t/q - D\mu_m \alpha^{-1}}{P_t/q + \mu_m \alpha^{-1}} x \left[1 - \frac{x}{[(\mu_m - D)P_t/(q\mu_m) - D\mu_m \alpha^{-1}]} \right]. \quad (1.12)$$

It clearly takes the form of the classical logistic equation

$$\frac{dx}{dt} = rx \left[1 - \frac{x}{K} \right] \quad (1.13)$$

with

$$r = \frac{(\mu_m - D)P_t/q - D\mu_m \alpha^{-1}}{P_t/q + \mu_m \alpha^{-1}}$$

and

$$K = [(\mu_m - D)P_t/(q\mu_m) - D\mu_m \alpha^{-1}].$$

It is easy to observe that both r and K are increasing functions of α and decreasing function of D . This makes good biological senses. It should be pointed out here that we did not assume the population suffers from a crowding effect explicitly. However, this crowding effect is implicitly provided by the fact that the total nutrient in the system (here P) is fixed, and individuals have to compete for this resource. Observe that instead of the often-assumed carrying capacity of the form P_t/q , here the carrying capacity has the expression of

$$K = [(\mu_m - D)P_t/(q\mu_m) - D\mu_m \alpha^{-1}]. \quad (1.14)$$

This says that although theoretically the environment may accommodate P_t/q plants, the actual upper limit the plant biomass can attain is $K = [(\mu_m - D)P_t/(q\mu_m) - D\mu_m \alpha^{-1}]$, less than that. The reason the maximal carrying capacity P_t/q cannot be reached in practice is that the death toll in a population keeps the population below its potential maximum. It says clearly that a population with a relatively

low death rate will likely amass more biomass than a population with a relatively high death rate.

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