

Lectures 26, Th., Nov. 15

1. Gause type predator-prey model and geometric criterion. We consider a class of Gause-type predator-prey model of the form

$$x' = xg(x) - yp(x), \quad y' = y(-d + q(x)), \quad x(0) \geq 0, \quad y(0) \geq 0, \quad (1.1)$$

where $\dot{} = d/dt$, and where p, q are sufficiently smooth so that solutions to initial value problems exist, are unique and are continuable for all positive t . We think of $x(t)$ and $y(t)$ as representing the prey and predator populations, respectively, at given time $t \geq 0$.

The following assumptions are consistent with models of predator-prey systems for $x, y \geq 0$.

(H1) : $g(0) > 0$; there exists $K > 0$ such that $g(x) > 0$ on $0 \leq x < K$, $g(K) = 0$, $g(x) < 0$ on $x > K$,

(H2) : $p(0) = 0, p'(x) > 0$.

(H3) : $q(0) = 0, q'(x) > 0$.

In most application, it is assumed that $q(x) = cp(x)$. The following simple stability result is straightforward from the Jacobian at E^* . It is often referred as the geometric criterion of Rosenzweig and MacArthur (1963), or simply the geometric criterion.

THEOREM 1.1. *Let $h(x) = xg(x)/p(x)$. In (1.1), assume that there is a $x^* < K$ such that $q(x^*) = d$. Let $E^* = (x^*, y^*)$ where $y^* = h(x^*)$. Then E^* is a sink if the slope of the x -isocline at E^* is negative (i.e. $h'(x^*) < 0$) and is unstable if the x -isocline at E^* is positive (i.e. $h'(x^*) > 0$).*

2. Holling type II predator-prey model and the paradox of enrichment.

If we assume that $g(x) = r(1 - \frac{x}{K}), p(x) = \frac{cx}{a+x}$ and $q(x) = \frac{bx}{a+x}$ in (1.1), we arrive at the following Holling type II predator-prey model.

$$x' = rx(1 - \frac{x}{K}) - y\frac{cx}{a+x}, \quad y' = y(-d + \frac{bx}{a+x}), \quad x(0) \geq 0, \quad y(0) \geq 0, \quad (2.1)$$

The following result is a direct application of Theorem 1.1.

THEOREM 2.1. *In (2.1), assume that $b > d$ and $x^* = \frac{ad}{b-d} < K$. Then $E^* = (x^*, y^*)$ where $y^* = h(x^*) = \frac{r}{c}(a+x)(1 - \frac{x}{K})$ is a sink if*

$$K < a + 2x^*.$$

If $K \gg 2x^$, then there is a large amplitude limit cycle surrounding E^* that can be very close to the axes.*

If we imagine the increasing of the carrying capacity of prey in the case of plant-insect interaction, the above theorem suggest that enrichment of the plant will lead to highly volatile insect population that naturally occurring stochastic event may wipe out the insect species (when the limit cycle is close to the x -axis) or both species (if the stochastic event wipe out the plant when the limit cycle is very close to the y -axis). This is termed as the *paradox of enrichment* (Rosenzweig, 1971). However, in reality, this is rarely observed.