Lectures 25, Tu., Nov. 13

The forth set of homework is postponed to Tuesday, November 13.

1. Applications of Liapunov-LaSalle Theorem in Population Models. Again we first recall the definition of Liapunov function. Let

$$x' = f(x), \;\; x \in R^n. \tag{1.1}$$

be an n-dimensional system of differential equations. Let $f(x)$ be defined on G^*, an open set in R^n, and let G be a subset of G^*. A function $V(x) : G \to R$ is said to be a *Liapunov function* for (1.1) on G if
1. V is continuously differentiable at each point $x \in G$, and
2. $\dot{V} = dV/dt|_{(1.1)} = \nabla \cdot V \leq 0$ on G.

The following Liapunov-LaSalle theorem will be the key in our effort in seeking global stability results in some population models.

THEOREM 1.1. *(Liapunov-LaSalle) Let V be a Liapunov function for (1.1) on a region G. Let $E = \{x|\dot{V}(x) = 0, x \in G \bigcap G^*\}$ and let M be the largest invariant set in E. Then every bounded (for $t \geq 0$) trajectory of (1.1) that remains in G tends to the set M as $t \to \infty$.*

Consider now the following Lotak-Volterra competition model

$$x' = x(r_1 - a_1x - b_1y), \;\; y' = y(r_2 - a_2x - b_2y). \tag{1.2}$$

We assume that (1.2)has a locally stable positive steady state $E^* = (x^*, y^*)$. This is equivalent to say that

$$\frac{a_1}{a_2} > \frac{r_1}{r_2} > \frac{b_1}{b_2} > 0. \tag{1.3}$$

Hence $a_1b_2 > a_2b_1$.

THEOREM 1.2. *In (1.2), if (1.3) holds, then all positive solutions tend to*

$$E^* = (x^*, y^*) = \left(\frac{r_1b_2 - r_2b_1}{a_1b_2 - a_2b_1}, \frac{r_2a_1 - r_1a_2}{a_1b_2 - a_2b_1}\right).$$

One can easily show that E is locally stable if the condition Again, we will proof the above theorem by applying Liapunov-LaSalle theorem. As in the previous example of the Lotka-Volterra predator-prey model, the key step is to construct an appropriate Liapunov function. To this end, we would like to try our luck for a Liapunov function that separate the variables x from y. In other word, we assume that

$$V(x, y) = V_1(x) + V_2(y).$$

To cut to the chase, we will assume that

$$V(x, y) = a_2(x - x^* \ln x) + b_1(y - y^* \ln y)$$

and

$$X = x - x^*, \;\; Y = y - y^*.$$

Then (1.2) can be rewritten as

$$x' = -x(a_1X + b_1Y), \; y' = -y(a_2X + b_2Y). \tag{1.4}$$

The derivative of this function along a solution of (1.2) takes the form of

$$\dot{V} = -[a_2a_1X^2 + 2a_2b_1XY + b_1b_2Y^2].$$

Notice that

$$(2a_2b_1)^2 < 4a_1a_2b_1b_2,$$

we have that

$$\dot{V} = -[a_2a_1X^2 + 2a_2b_1XY + b_1b_2Y^2] \le 0$$

and $\dot{V} = 0$ iff $(x, y) = E^*$. This shows that $E = \{E^*\} = M$. An application of the Liapunov-LaSalle theorem shows that all positive solutions of (1.2) tend to the unique positive steady state E^*.

We have covered the Routh-Hurwitz criteria with focus on the case of $k = 3$.

THEOREM 1.3. *(Routh-Hurwitz criteria for third order polynomials) If* $a_1 > 0, a_3 > 0$ *and* $a_1a_2 > a_3$ *in*

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{1.5}$$

then all the roots of (1.5) have negative real parts.

In particular, we covered Example 1 on page 234 (not in detail, ***so please read it carefully***). Assume that the two prey-one predator model (21) in the textbook Example 1 on page 234 has a positive equilibrium $E^* = (x^*, y^*, z^*)$. Using a Liapunov function of the form with appropriate values of a, b and c (such as $a = \chi/\alpha, b = \beta\chi/(\alpha\varepsilon)$ and $c = 1$)

$$V = a(x - x^* \ln x) + b(y - y^* \ln y) + c(z - z^* \ln z),$$

we have shown (with some interesting work) that the positive steady state

$$E^* = (\frac{\delta}{\varepsilon}, \gamma - \alpha(\nu - \frac{\delta\chi}{\varepsilon\mu}), \nu - \frac{\delta\chi}{\varepsilon\mu})$$

is globally stable.