Lectures 24, Th., Nov. 8

Reading homework: Chapter 6

The forth set of homework is postponed to Tuesday, November 13.

1. Applications of Liapunov-LaSalle Theorem in a Food Chain, II. We first recall the definition of Liapunov function. Let

$$x' = f(x), \;\; x \in R^n. \tag{1.1}$$

be an n-dimensional system of differential equations. Let $f(x)$ be defined on G^*, an open set in R^n, and let G be a subset of G^*. A function $V(x) : G \to R$ is said to be a *Liapunov function* for (1.1) on G if

1. V is continuously differentiable at each point $x \in G$, bounded from below in G, and
2. $\dot{V} = dV/dt|_{(1.1)} = \nabla \cdot V \leq 0$ on G.

The following Liapunov-LaSalle theorem will be the key in our effort in seeking global stability results in some population models.

THEOREM 1.1. *(Liapunov-LaSalle) Let V be a Liapunov function for (1.1) on a region G. Let $E = \{x|\dot{V}(x) = 0, x \in G \bigcap G^*\}$ and let M be the largest invariant set in E. Then every bounded (for $t \geq 0$) trajectory of (1.1) that remains in G tends to the set M as $t \to \infty$.*

Consider now the following Lotak-Volterra food chain model

$$x' = x[r(1-x/K)-by] = x(r-ax-by), \;\; y' = y(-m+cx-dz), \;\; z' = z(ey-f). \tag{1.2}$$

We assume that (1.2)has a positive steady state $E = (x^*, y^*, z^*)$. This is equivalent to say that $re > bf$ and $\frac{c}{a}(r - b\frac{f}{e}) > 0$.

THEOREM 1.2. *In (1.2), if $re > bf$ and $\frac{c}{a}(r - b\frac{f}{e}) > 0$, then all positive solutions tend to $E = (\frac{1}{a}(r - b\frac{f}{e}, \frac{f}{e}, \frac{c}{da}(r - b\frac{f}{e})$.*

Again, we will proof the above theorem by applying Liapunov-LaSalle theorem. As in the previous example of the Lotka-Volterra predator-prey model, the key step is to construct an appropriate Liapunov function. To this end, we would like to try our luck for a Liapunov function that separate the variables x, y and z. In other word, we assume that

$$V(x, y, z) = V_1(x) + V_2(y) + V_3(z).$$

Let

$$X = x - x^*, \;\; Y = y - y^*, \;\; Z = z - z^*.$$

Then (1.2) can be rewritten as

$$x' = x(-aX - bY), \;\; y' = y(cX - dZ), \;\; z' = z(eY). \tag{1.3}$$

The derivative of this function along a solution of (1.2) takes the form of

$$\dot{V} = V_1'(x)x(-aX - bY) + V_2'(y)y(cX - dZ) + V_3'(z)zeY$$

$$= -axXV_1'(x) - bV_1'(x)xY + V_2'(y)ycX - V_2'(y)ydZ + V_3'(z)zeY.$$

We would like to have all the mixed terms in the expression of $\dot{V}$ to cancel each other. In other words, we want

$$bV_1'(x)xY = V_2'(y)ycX,\ V_2'(y)ydZ = V_3'(z)zeY.$$

This is equivalent to say that

$$f(x) \equiv bV_1'(x)x/X = V_2'(y)yc/Y \equiv g(y)$$

and

$$V_2'(y)yd/Y \equiv \frac{d}{c}g(y) = V_3'(z)ze/Z \equiv= h(z).$$

This says a function of x is identical to a different function of another independent variable y and a function of y is identical to a different function of another independent variable z. This can only be true if these functions are constants. For our purpose, we may simply assume that the first constant is 1 ($= f(x) = g(y)$). This yields

$$bV_1'(x) = X/x = 1 - x^*/x,\ cV_2'(y) = Y/y = 1 - y^*/y.$$

and

$$\frac{ce}{d}V_3'(z) = Z/z = 1 - z^*/z.$$

This is turn suggest that

$$V_1(x) = \frac{1}{b}(x - x^* \ln x),\ V_2(y) = \frac{1}{c}(y - y^* \ln y),\ V_3(z) = \frac{d}{ce}(z - z^* \ln z).$$

With these functions, we have

$$\dot{V} = -axXV_1'(x) = -\frac{a}{b}(x - x^*)^2.$$

Hence, $V = \frac{1}{b}(x - x^* \ln x) + \frac{1}{c}(y - y^* \ln y) + \frac{d}{ce}(z - z^* \ln z)$ is indeed a Liapunov function for (1.2). We have $E = \{(x, y, z) : x = x^*, y, z \geq 0\}$. We shall show that the largest invariant set M in E for (1.2) is $\{E\}$. To this end, we assume that $(x(0), y(0, z(0)) \in M$. Since $(x(t), y(t), z(t)) \in E$, we have $x(t) \equiv x^*$ which implies that

$$x'(t) \equiv 0.$$

This in turn yields

$$0 = x'(t) = x^*(r - ax^* - by(t))$$

and hence $y(t) = y^*$. In particular, we must have $y(0) = y^*$. Similarly, we must have $z(0) = z^*$, proving that $M = \{E\}$. This shows that all positive solutions of (1.2) tend to the unique positive steady state E.

It can be shown that

$$\min\{V(x, y, z) : x > 0, y > 0, x > 0\} = V(x^*, y^*, z^*).$$

As a by product of this and above global stability result, we see that

$$V(x,y,z) \le V(x(0),y(0),z(0))$$

which is equivalent to

$$V(x,y,z) - V(x^*,y^*,z^*) \le V(x(0),y(0),z(0)) - V(x^*,y^*,z^*) = v,$$

where $v \ge 0$. Hence $x - x^* - x^* \ln(x/x^*) < bv$, implying x is bounded. Similarly, we see that y and z are also bounded.

We provid below an alternative, elementary, and somewhat more tedious proof of the boundedness of the solution without the need of a Liapunov function nor the need of the existence of a positive steady state. This mimics the approach of the boundedness proof for the chemostat model.

Let

$$M(t) = x(t) + Ay(t) + Bz(t).$$

We have

$$M'(t) = x(r-ax) - bxy - mAy + cAxy - dAyz + eBzy - fBz$$

. Let $A = \frac{b}{c},\ B = \frac{db}{ce}$, then

$$M'(t) = x(r+m-ax) - mx - \frac{mb}{c}y - \frac{fdb}{ce}z.$$

Clearly $x(r+m-ax) \le (r+m)^2/(4a) \equiv l$. Let $k = \min\{1, f/m\}$, then we have

$$M'(t) \le l - mkM(t).$$

This implies that

$$M(t) \le \frac{l}{mk} + (M(0) - \frac{l}{mk})e^{-mkt} \le \max\{M(0), l/mk\}.$$