

Lectures 15, Tu., October 9, 2007

Reading homework: Chapter 5

1. A Droop equation based derivation of the Monod function. Recall the basic chemostat model with its original parameters

$$\begin{cases} N' &= K(C)N - \frac{FN}{V}, \\ C' &= -\alpha K(C)N - \frac{FC}{V} + \frac{FC_0}{V}. \end{cases} \quad (1.1)$$

On page 125, the Monod function $\frac{K_{max}C}{K_n + C}$ (called Michaelis-Menten function there and often referred as Holling type II function in population biology) is selected for $K(C)$ without justification. An excellent account of a classical derivation can be found in the classical book of Murray (p 175, chapter 6, Mathematical Biology I, 2001). In the following, we present a novel derivation based on Droop equation.

Recall that the Droop equation takes the form of

$$\mu(Q) = \mu_{max} \left(1 - \frac{q}{Q}\right). \quad (1.2)$$

Hence, we can think $K(C) = \mu(Q)$. The rate of cell nutrient loss due to growth is $\mu(Q)Q = \mu_{max}(Q - q)$. Assume that the bacterial cell has a nutrient uptake rate of ρ and the growth, then we have

$$Q' = \rho C - \mu_{max}(Q - q). \quad (1.3)$$

The cell quote dynamics is much faster than growth dynamics. An application of the quasi-steady-state argument by setting $Q' = 0$ yield

$$Q = q + \frac{\rho}{\mu_{max}}C.$$

Hence

$$K(C) = \mu(Q) = \mu_{max} \left(1 - \frac{q}{q + \frac{\rho}{\mu_{max}}C}\right) = \frac{\mu_{max}C}{q\mu_{max}/\rho + C}.$$

We see that $K_{max} = \mu_{max}$ and $K_n = q\mu_{max}/\rho$.

2. A basic chemostat model: optimum flow rate. In application, a chemostat can be used to produce bacteria. The bacteria is harvested continuously by collecting the bacteria contained in the outflow. In the following, we will answer the question what is the optimum flow rate to ensure maximum bacteria yield for the following chemostat model.

$$\begin{cases} N' &= \frac{K_{max}CN}{K_n + C} - \frac{FN}{V}, \\ C' &= -\alpha \frac{K_{max}CN}{K_n + C} - \frac{FC}{V} + \frac{FC_0}{V}. \end{cases} \quad (2.1)$$

We know now that the positive steady state

$$E^* = (N^*, C^*) = \left(\frac{C_0 - C^*}{\alpha}, \frac{FK_n}{VK_{max} - F} \right)$$

exists when $K_{max} > F/V$ and $C_0 > C^* = \frac{FK_n}{VK_{max} - F}$. This amounts to assume that

$$F < \frac{C_0VK_{max}}{C_0 + K_n}.$$

E^* is globally stable whenever it exists. Hence we will assume below that E^* exists and the chemostat is operating at the steady state. In this scenario, the yield rate is

$$Y(F) = FN^* = F \frac{C_0 - C^*}{\alpha} = \frac{F(C_0VK_{max} - (C_0 + K_n)F)}{\alpha(VK_{max} - F)}.$$

The optimum yield will be achieved at the value of F such that $Y'(F) = 0$. This together with the restriction that $F < \frac{C_0VK_{max}}{C_0 + K_n}$ leads to the optimum flow rate of

$$F^* = VK_{max} \left(1 - \sqrt{\frac{K_n}{C_0 + K_n}} \right).$$