

Lectures 13-14, Tu., October 2-4, 2007

Reading homework: Chapter 5

1. A basic chemostat model. On Tuesday, Oct. 2, we have covered in detail the section 4.5 on dimensional analysis of the chemostat model. As a result, we arrive at the following simplified basic chemostat model

$$\begin{cases} N' &= \alpha_1 \frac{CN}{1+C} - N, \\ C' &= -\frac{CN}{1+C} - C + \alpha_2. \end{cases} \quad (1.1)$$

In addition, we have covered the preliminary analysis of the basic chemostat model on the positivity and boundedness of its solutions with respect to positive initial values.

On Thursday, Oct. 4, we will cover the sections 4.9-4.10 in details. In addition, we will present two global stability results.

2. Basic chemostat model: preliminary analysis. We shall show below that the solution of (1.1) with $N(0) > 0$ and $0 \leq C(0) \leq \alpha_2$ are positive and bounded componentwise.

THEOREM 1. (*Positivity of the basic chemostat model*) *If $N(0) > 0$ and $0 \leq C(0) \leq \alpha_2$, then for all $t > 0$, $0 < N(t) \leq \max\{N(0), \alpha_1\alpha_2\}$ and $0 < C(t) < \alpha_2$. solutions of (1.1) with $N(0) > 0$ and $0 \leq C(0) \leq \alpha_2$ are positive and bounded componentwise.*

Proof. We will first establish the positivity of the solution. Notice that if $C(0) = 0$, then $C'(0) = \alpha_2 > 0$ and by continuous dependence of the solution and its derivative on time, we see that there is a $t_0 > 0$ such that $C(t) > 0$ for $t \in (0, t_0]$.

Assume that there is a $t_1 > 0$ such that $N(t_1)C(t_1) = 0$, $C(t) > 0$ and $N(t) > 0$ for $t \in (0, t_1)$. We have two cases to consider: 1) $N(t_1) = 0$; 2) $C(t_1) = 0$. Consider first that $N(t_1) = 0$. The first equation of (1.1) gives

$$N'(t) \geq -N(t),$$

for $t \in (0, t_1]$. This implies that $N(t) \geq N(0)e^{-t} > 0$. A contradiction indicating that case 1) is impossible. Consider now the case that $C(t_1) = 0$. Clearly, $C(t_1) = 0$ and $C(t) > 0$ for $t \in (0, t_1)$ implies that $C'(t_1) \leq 0$. However, the second equation of (1.1) gives $C'(t_0) = \alpha_2 > 0$. This clearly is a contradiction which indicates that case 2) is impossible. This establishes the positivity.

We now turn to boundedness. It is easy to see that $C(0) \leq \alpha_2$ for $t > 0$. If not, then by mean value theorem, there is a $t_2 > 0$, such that $C(t_2) > \alpha_2$ and $C'(t_2) > 0$. However, this is impossible from the the second equation of (1.1). Let $Z = N + \alpha_1 C$. Then

$$Z'(t) = \alpha_1\alpha_2 - Z(t).$$

Hence

$$Z(t) = \alpha_1\alpha_2 + (Z(0) - \alpha_1\alpha_2)e^{-t} = \alpha_1\alpha_2(1 - e^{-t}) + Z(0)e^{-t} \leq \max\{N(0), \alpha_1\alpha_2\}.$$

Therefore $0 < N(t) \leq \max\{N(0), \alpha_1\alpha_2\}$ for all $t > 0$. ■

3. Global stability results for the chemostat model. We shall present two global stability results for the chemostat model (1.1).

THEOREM 2. (*Global stability of $E_2 = (\bar{N}_2, \bar{C}_2)$)* If $C(0) < \alpha_2$, $\alpha_1 > 1$ and $\alpha_2 < \frac{1}{\alpha_1 - 1}$, then $\lim_{t \rightarrow \infty} (N, C) = (\bar{N}_2, \bar{C}_2)$.

Proof. We show first that $C(t) < \alpha_2$ for $t > 0$. If not, then there is a $t_1 > 0$, such that $C(t_1) = \alpha_2$ and $C'(t_1) \geq 0$. However from the C equation, we see that $C'(t_1) = -\frac{C(t_1)N(t_1)}{1 + C(t_1)} < 0$, a contradiction. Observe that $\alpha_2 < \frac{1}{\alpha_1 - 1}$ implies that $\alpha_2\alpha_1 < 1 + \alpha_2$. Let $\theta = \alpha_2\alpha_1/(1 + \alpha_2)$. Then we have

$$N' = \alpha_1 \frac{CN}{1 + C} - N < -(1 - \theta)N.$$

This shows that $\lim_{t \rightarrow \infty} N = 0$. In particular, for any $\varepsilon > 0$, there is a $T > 0$ such that $N(t) < \varepsilon$ for $t > T$. Hence for $t > T$, $-C + \alpha_2 - \varepsilon < C' < -C + \alpha_2$. Standard comparison arguments (compare to $C' = -C + \alpha_2$ and $C' = -C + \alpha_2 - \varepsilon$) yield that for large $t > T$, $C(t) \in [\alpha_2 - \varepsilon, \alpha_2]$. Letting $\varepsilon \rightarrow 0$ yields that $\lim_{t \rightarrow \infty} (N, C) = (\bar{N}_2, \bar{C}_2)$. ■

THEOREM 3. (*Global stability of $E_1 = (\bar{N}_1, \bar{C}_1)$)* If $C(0) < \alpha_2$, $\alpha_1 > 1$ and $\alpha_2 > \frac{1}{\alpha_1 - 1}$, then $\lim_{t \rightarrow \infty} (N, C) = (\bar{N}_1, \bar{C}_1)$.

Proof. Let $Z(t) = N(t) + \alpha_1 C(t)$. Then

$$Z'(t) = \alpha_1 \alpha_2 - Z(t).$$

Hence $Z(t) = \alpha_1 \alpha_2 + (Z(0) - \alpha_1 \alpha_2)e^{-t}$. This shows that $Z(t)$ tends to $\alpha_1 \alpha_2$ monotonically. Hence the global dynamics is the same as that of (by a result of H. R. THIEME, *Mathematics in Population Biology*, Princeton Series in Theoretical and Computational Biology, Princeton University Press (2003)) the limiting system of $N(t) = \alpha_1(\alpha_2 - C(t))$ and $N' = -\alpha_1 C' = \alpha_1 \left(\frac{\alpha_1 C}{1 + C} - 1 \right) \alpha_1 (\alpha_2 - C(t)) = \alpha_1 \frac{(\alpha_1 - 1)(C - \bar{C}_1)}{1 + C} (\alpha_2 - C(t))$. It is easy to see that $\lim_{t \rightarrow \infty} C = \bar{C}_1$. This implies that $\lim_{t \rightarrow \infty} N = \bar{N}_1$. ■