MAT 494: Mathematical Models in Biosciences

Solution to exercise 9 on page 104

Consider the following Nicholson-Bailey host-parasitoid model

$$\begin{cases} N_{t+1} &= N_t e^{r(1-N_t/K)-aP_t} \equiv F(N_t, P_t), \\ P_{t+1} &= N_t(1-e^{-aP_n}) \equiv G(N_t, P_t). \end{cases} \tag{1}$$

Let $E = (\bar{N}, \bar{P})$ be a positive equilibrium of (1) and $q = \bar{N}/K$. We have Consider the following Nicholson-Bailey host-parasitoid model

$$\begin{aligned} \bar{N} &= \bar{N} e^{r(1-\bar{N}/K)-a\bar{P}}, \\ \bar{P} &= \bar{N}(1-e^{-a\bar{P}}). \end{aligned} \tag{2}$$

The first equation of (2) implies that

$$\bar{P} = \frac{r}{a}(1-\frac{\bar{N}}{K}) = \frac{r}{a}(1-q). \tag{3}$$

The second equation of (2) implies that

$$\bar{N} = \frac{\bar{P}}{1-e^{-a\bar{P}}}. \tag{4}$$

The Jacobian at E takes the form

$$A = \begin{pmatrix} \partial F(\bar{N}, \bar{P})/\partial N_t & \partial F(\bar{N}, \bar{P})/\partial P_t \\ \partial G(\bar{N}, \bar{P})/\partial N_t & \partial G(\bar{N}, \bar{P})/\partial P_t \end{pmatrix} = \begin{pmatrix} 1-rq & -a\bar{N} \\ \bar{P}/\bar{N} & a\bar{N}e^{-a\bar{P}} \end{pmatrix}.$$

Hence we hace

$$\beta = 1 - rq + a\bar{N}e^{-a\bar{P}} = 1 - rq + a\bar{N}e^{-r(1-q)} = 1 - rq + a\frac{\bar{P}e^{-r(1-q)}}{1-e^{-a\bar{P}}}.$$

Using (3), we obtain

$$\beta = 1 - rq + \frac{r(1-q)e^{-r(1-q)}}{1-e^{-r(1-q)}} = 1 - r + \frac{r(1-q)}{1-e^{-r(1-q)}}.$$

If we let

$$\phi = r(1-q)/[1-e^{-r(1-q)}].$$

Then

$$\beta = 1 - r + \phi.$$

Clearly, we have

$$\gamma = (1-rq)a\bar{N}e^{-a\bar{P}} + a\bar{P}.$$

Thus

$$\gamma = r(1-q) + (1-rq)r(1-q)e^{-r(1-q)}/(1-e^{-r(1-q)}) = r^2q(1-q) + (1-rq)\phi.$$

So the equilibrium E is stable if

$$|\beta| < \gamma + 1 < 2. \tag{5}$$

The stability region is bounded by the curves of $|\beta| = \gamma + 1$ and $\gamma = 1$.

The following is the matlab file that generates the Figure 1.

```
%Stability boundary for the Nicholson-Bailey host-parasitoid model.
figure
r=0:0.025:5; q=0:0.005:1;
a=0.4; d=0.01; c=0.1;
[R,Q]=meshgrid(r,q);
phi=R.*(1-Q)./(1-exp(-R.*(1-Q)));
l1=(-1+R-phi).*(-1+R-phi)-
((1-R.*Q).*phi+R.*R.*Q.*(1-Q)+1).*((1-R.*Q).*phi+R.*R.*Q.*(1-Q)+1);
l2=(1-R.*Q).*phi+R.*R.*Q.*(1-Q)-1;
cvals=[0 0];
hold on
contour(r,q,l1,cvals,'r', 'LineWidth',2);
contour(r,q,l2,cvals,'k');
legend('\it |\beta|=\gamma +1','\gamma=1',4),
xlim([0 5]), ylim([0 1]),
```

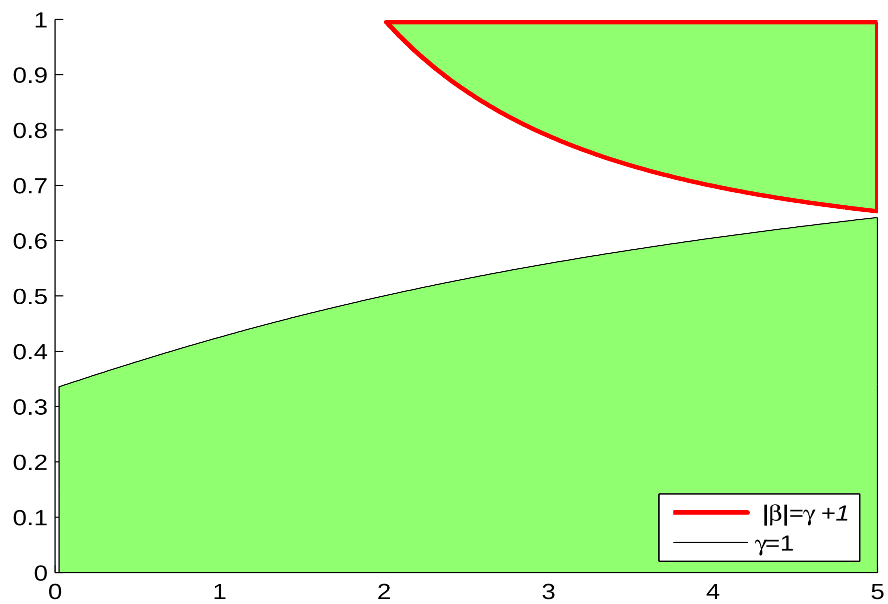

Figure 1: Reproduction of Figure 3.4 on page 84. The density dependent Nicholson-Bailey model is stable in the white area.