SAMPLE Math Bio Qualifier

Nov. 28, 2006

Answer 4 out of 6.

1. A population starts with 1000 members of age zero. Assume that each member of age zero produces two offspring and that 2/5 survive to age one. All members of age one produce 2 offspring and then die.
   (a) Find the corresponding Leslie matrix, its dominant eigenvalue and a corresponding eigenvector.
   (b) Describe the stable age distribution and the asymptotic behavior of the population.

2. Consider the model

$$x_{n+1} = ax_n \exp(-x_n), \qquad x_0 > 0. \tag{1}$$

   (a) Show that for $n = 1, 2, 3...$, $x_n \leq \max\{x_0, a/e\}$.
   (b) Show that $\lim_{n\to\infty} x_n = 0$ if $a \leq 1$.
   (c) Assume $1 < a < e^2$. Show that the positive steady state $x_n \equiv \ln a$ is locally stable.
   (d) Assume $1 < a < e$. Show that the positive steady state $x_n \equiv \ln a$ is globally stable.
   (e) (bonus) Assume $e < a < e^2$. Show that the positive steady state $x_n \equiv \ln a$ is globally stable.

3. Consider the following simple discrete-time SI model

$$S_{n+1} = S_n e^{-aI_n/(S_n+I_n)}, \quad I_{n+1} = I_n + S_n(1 - e^{-aI_n/(S_n+I_n)}),\ S_0 > 0,\ I_0 > 0. \tag{2}$$

   (a) What are S_n, I_n and a biologically. Explain the model formulation.
   (b) Show that solutions are positive and bounded. What is R_0 for this model?
   (c) Show that $\lim_{n\to\infty}(S_n, I_n) = (S^*, 0)$ for some $S^* \in [0, S_0]$.

4. Consider the following predator-prey system

$$\begin{aligned} x' &= r(1 - x/K - cy),\ x(0) > 0 \\ y' &= y(-d + x - fy),\ y(0) > 0 \end{aligned}$$

with $r, K, c, d > 0$ and $f \geq 0$.
(a) Show that the solution is positive and bounded.
(b) Assume that $K > d$. Find all the nonegative steady states and determine their local stabilities.
(c) Show that the positive steady state is globally asymptotically stable.

5. Consider a population that is divided into three groups by an infectious disease, say susceptibles, infectives and removals. Let $S(t), I(t)$ and $R(t)$ denote the numbers in these populations at time t. An variation of the Kermack-McKendrick model takes the form of

$$\begin{aligned} \dot{S} &= -aI\frac{S}{S+I+R} + bR \\ \dot{I} &= aI\frac{S}{S+I+R} - cI - dI \\ \dot{R} &= cI - bR. \end{aligned}$$

a): Explain the formulation of the above model. What are the meanings of the parameters a, b, c and d.
b): If $d > 0$, show that no endemic equilibrium exist and the disease always eventually disappear.
c): Assume $d = 0$. Show that the total population stay constant. What are the biological implications of the conditions for the existence of an endemic equilibrium (positive steady state). Do you need a minimum total population level to have such an endemic equilibrium?

6. Consider the following one-prey two-predators model with both predators exhibiting self-crowding effect,

$$\begin{aligned} x' &= x(5 - 2x - y - 2z), \\ y' &= y(-2 + 3x - y), \\ z' &= z(-1 + 2x - z). \end{aligned}$$

a): Show that this model possesses an unique positive steady state and find the stability of this positive steady state. (hint: use Routh–Hurwitz criteria).
b): Apply the Liapunov-LaSalle theorem to show that this positive steady state is globally asymptotically stable. Comment on the biological implications of this mathematical finding. (hint: what is competitive exclusion principle and why it fails here?)